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O. Neugebauer J. D. Tamarkin O. Veblen

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ANALYSIS

Calculus

***de Losada y Puga, Cristóbal. Course of Mathematical Analysis.** Vol. 1. Universidad Católica del Perú, Lima, 1945. xxvi+632 pp. (Spanish)

The Spanish title is Curso de Análisis Matemático. The topics covered in this volume are: Numbers, Variables, Functions, Limits, Derivatives, Integrals, Introduction to Differential Equations.

Mahajani, G. S. A note on Riemann integration. J. Univ. Bombay (N.S.) 10, part 3, 9–11 (1941). [MF 12185]

Evaluation of integrals of elementary functions, like x^n, starting from the definition.

Fraile, Arturo. Differentiation and integration of absolute values of functions of a real variable. Revista Union Mat. Argentina 10, 84–92 (1945). (Spanish) [MF 12278]

Fort, Tomlinson. Taylor's formula and Sterling's numbers. Nat. Math. Mag. 19, 163–170 (1945). [MF 12320]

Pólya, G. Approximations to the area of the ellipsoid. Publ. Inst. Mat. Univ. Nac. Litoral 5, 13 pp. (1943). [MF 12473]

Several approximate formulas for the area of an ellipsoid with semiaxes a, b, c are stated, together with inequalities for the sign of the error and estimates of the error for the case where b and c are nearly equal to a. Some of the results are new, but no detailed proofs are included.

P. Franklin (Cambridge, Mass.).

Gladwin, A. S. A note on a certain multiple integral. Philos. Mag. (7) 35, 657–660 (1944). [MF 11955]

The formula obtained is

$$n!\int_a^a f(u_1)du_1\int_{u_1}^a f(u_2)du_2\cdots\int_{u_{n-1}}^a f(u_n)du_n=\left\{\int_a^a f(u)du\right\}^n.$$

[This formula is not new and can be proved in a few lines by induction.] *R. P. Boas, Jr.* (Providence, R. I.).

Horenstein, William. On certain integrals in the theory of heat conductions. Quart. Appl. Math. 3, 183–184 (1945). [MF 12657]

$$\varphi\equiv\int_0^t x^{-1}\exp(-a^2x^{-1}-b^2x)dx=\tfrac{1}{2}\pi^{\frac{1}{2}}a^{-1}\cosh 2ab$$
$$+\tfrac{1}{4}\pi^{\frac{1}{2}}a^{-1}\{e^{-2ab}\operatorname{erf}(bc^{-1}-ac)-e^{2ab}\operatorname{erf}(bc^{-1}+ac)\}$$
$$\left(\operatorname{erf}(x)=2\pi^{-\frac{1}{2}}\int_0^x e^{-t^2}dt\right);$$
$$\int_0^t x^{-1}\exp(-a^2x^{-1}-b^2x)dx=-\tfrac{1}{2}b^{-1}d\varphi/db.$$

R. P. Boas, Jr. (Providence, R. I.).

Differential Equations

Noronha, F. J. On differential equations reducible to Clairaut's form. Math. Student 12, 70 72 (1945). [MF 12484]

Peñalver, P. On a particular class of linear homogeneous differential equations of order m. Revista Mat. Hisp.-Amer. (4) 3, 357–360 (1943). (Spanish) [MF 12177]

This elementary paper gives a necessary and sufficient condition for a homogeneous linear differential equation, of order $m\geqq k$, to possess the particular solutions $x, x^2, \cdots, x^k$.

L. A. MacColl (New York, N. Y.).

Lahaye, Edmond. Sur la résolution des équations $dz/dx=R(x,z)$ (R rationnel en z) par des itérations intégrales et différentielles convergentes. J. Math. Pures Appl. (9) 22, 1–23 (1943). [MF 12248]

Consider the differential equation in the complex domain

$$\frac{dw}{dz}=\frac{\sum_{k=0}^{p+1}C_k(z)w^k}{p+\sum_{k=1}^{p-1}D_k(z)w^k} \tag{1}$$

and let L be an arc of finite length in the z plane which encounters no singularities of the equation (1). The Picard method of successive approximations applied to (1) will not in general give a solution holding all along the arc L, while the Cauchy-Lipschitz method may not hold in the domain of the movable singularities of (1). This paper gives two methods of successive approximations which give a solution throughout the length of L and also converge in the domain of the movable singularities of equation (1).

F. G. Dressel (Durham, N. C.).

Wintner, Aurel. The non-local existence problem of ordinary differential equations. Amer. J. Math. 67, 277–284 (1945). [MF 12433]

The author considers the system of differential equations $\dot{x}_i=f_i(x_1,\cdots,x_n)$, where f_i are real-valued and continuous for all values of x_k and

$$(*)\qquad f_i(x_1,\cdots,x_n)=O(|x|),\qquad |x|\to\infty,$$

$|x|=\max|x_i|$. This system has a solution $x_i(t)$, corresponding to any "initial conditions" $x_i(0)=c_i$, which exists on the whole t-axis, $-\infty<t<\infty$. Moreover, $x_i(t)=O(e^{ct})$ as $t\to\infty$, where c is a constant. It is shown that (*) can be weakened by considering logarithmic terms along with $|x|$ itself, but that the integral to ∞ of the reciprocal of the function used in the right of (*) must diverge. Extensions are made to the case where f_i involves t.

N. Levinson (Cambridge, Mass.).

Četajev, N. G. Theorem concerning the non-stability of regular systems. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 323–326 (1944). (Russian. English summary) [MF 12231]

The author shows that, if at least one of the characteristic numbers associated with the system of differential equations

$\dot{x}_i = X_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$, is positive, then in a certain sense the system is unstable. *N. Levinson.*

Malkin, I. Stability of periodic motions of dynamic systems. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 327–331 (1944). (Russian. English summary) [MF 12232]

The author considers the system

$$(*) \qquad \dot{z}_i = Z_i(t, z_1, \cdots, z_n), \qquad i = 1, \cdots, n,$$

where the Z_i are of class C_2 with respect to the z_k and are periodic in t with period T. Let $z_i = \phi_i(t, h_1, \cdots, h_m)$, $m < n$, be a solution of (*), where the ϕ_i are periodic in t of period T and $h_1, \cdots, h_n$ are arbitrary constants. In this case m of the characteristic numbers associated with the solution vanish. If the remaining $n-m$ have negative real parts then the solution is stable in the sense of Liapounoff.

N. Levinson (Cambridge, Mass.).

Horn, J. Integration linearer Differentialgleichungen durch Laplacesche Integrale. I. Math. Z. **49**, 339–350 (1944). [MF 11975]

It is known that a system of complex linear differential equations of the first order can be solved by ordinary Laplace integrals if it is of rank unity; for a system of rank two, solutions have been given by use of generalized Laplace integrals [see, for example, Ince, Ordinary Differential Equations, Longmans, Green, London, 1927, pp. 479, 481]. The author obtains conditions under which a system of rank two can be solved by use of ordinary Laplace integrals. As a special case of his result it follows that a system of rank two, consisting of two or three simultaneous equations, always admits such a solution. [Cf. the following review.]

H. Pollard (New York, N. Y.).

Horn, J. Integration linearer Differentialgleichungen durch Laplacesche Integrale. II. Math. Z. **49**, 684–701 (1944). [MF 11976]

The author continues the investigations begun in part I [see the preceding review]. It is shown that a single complex differential equation of order m, with an irregular point of rank two at infinity, can under certain conditions be broken down into a special system of m equations of the first order; in this form it is amenable to solution by ordinary Laplace integrals. *H. Pollard* (New York, N. Y.).

Pierce, Jesse. Solutions of systems of differential equations in the vicinity of branch points of the solutions. III. Duke Math. J. **12**, 37–41 (1945). [MF 12068]

[The first two papers appeared in the same J. **4**, 650–655 (1938); **11**, 83–88 (1944); these Rev. **5**, 183.] The paper is concerned with the system of differential equations

$$x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} dx_i/dt = a_i/(m+1) + f_i(t) + \sum_{\nu=1}^{\infty} f_{i\mu}^{(\nu)} x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}, \qquad i = 1, \cdots, n,$$

where m is a positive integer, $\mu_1, \mu_2, \cdots, \mu_n$ are nonnegative integers, μ represents the sequence $\mu_1, \mu_2, \cdots, \mu_n$, $\nu = \mu_1 + \mu_2 + \cdots + \mu_n$, $a_i \neq 0$ and $\sum_{j=1}^{n} \alpha_{ij} = m$, $i = 1, \cdots, n$. Conditions of boundedness and integrability too intricate to be listed here are assumed for the coefficients $f_i(t)$ and $f_{i\mu}^{(\nu)}$. By obtaining formal solutions and dominant series, it is shown that, if the f_i and $f_{i\mu}^{(\nu)}$ can be expressed in power series in $t - t_0$, the x_i can be expressed as power series in $(t-t_0)^{1/(m+1)}$. There are in general $m+1$ distinct solutions with $x_i(t_0) = 0$. The case $\alpha_{ij} = 0$, $j \neq i$; $\alpha_{ii} = m$; $a_i = 1$ was treated in part II.

R. E. Langer (Madison, Wis.).

Malmquist, J. Sur les points singuliers des équations différentielles. Ark. Mat. Astr. Fys. **29A**, no. 18, 11 pp. (1943). [MF 12019]

The author states, without proof, a number of results which are complementary to those given in a series of earlier papers relating to the solutions of systems of differential equations of the form

$$x^{h+1} y_i' = P_i(x, y_1, \cdots, y_n), \qquad i = 1, \cdots, n,$$

where h is a positive integer and the P's are power series which vanish for $x = y_1 = \cdots = y_n = 0$. The new results, which cannot be stated briefly in an explicit way, relate to the reduction of the system of differential equations to a form which is convenient for the study of the solutions in the neighborhood of the point $x = 0$. They represent improvements on the corresponding results given in the earlier papers in that some features of the analysis are simplified and the regions of convergence of certain series are more definitely determined. *L. A. MacColl.*

Malmquist, J. Sur un système d'équations différentielles étudié par M. Störmer. Ark. Mat. Astr. Fys. **30A**, no. 5, 8 pp. (1944). [MF 12005]

This paper relates to the trajectories of an electrified particle moving in the field of a magnetic dipole. Numerical solutions of the differential equations by C. Störmer made it seem very probable that there exist trajectories which pass through the location of the dipole. In the present paper the existence of a class of such trajectories is demonstrated analytically. Whether or not the entire class of such trajectories has been found is left undecided. The demonstration depends in part on earlier work by the author on the solutions of systems of differential equations of the form discussed in the paper reviewed above. *L. A. MacColl.*

Pougatcheff, V. On asymptotic representation of integrals of systems of linear differential equations containing a parameter. Rec. Math. [Mat. Sbornik] N.S. **15** (57), 13–54 (1944). (Russian. English summary) [MF 12280]

This is a substantial development of the asymptotic theory of differential systems containing a complex parameter α, in the case when the characteristic roots are distinct for every x on a closed interval (a, b). Capital letters are used for matrices. Asymptotic relations are to a finite (always specified) number of terms in a domain extending to ∞. The order of differentiability is also always specified. The system studied is

$$(1) \qquad Y' = YP + \Pi, \quad P \sim \alpha^r (P^{(0)} + P^{(1)} \alpha^{-1} + \cdots), \quad \Pi \sim \alpha^s Z (\Pi^{(0)} + \cdots)$$

(r, s integers); Z is a solution of

$$(2) \qquad Z' = ZQ, \quad Q \sim \alpha^r (Q^{(0)} + Q^{(1)} \alpha^{-1} + \cdots).$$

The characteristic roots are those of $P^{(0)}$. By a transformation $Y = U \sum_0^{\infty} S_\nu \alpha^{-\nu}$ (S_0 reducing $P^{(0)}$ to diagonal form) the system (3) $Y' = YP$ is reduced to another system involving a diagonal matrix. In this connection certain quantities $\lambda_i(x, \alpha)$ are introduced, termed the characteristic polynomials of (3). The supposition is made that every $\Re(\lambda_i - \lambda_j)$ is bounded above or below. The main results are as follows. If (3) and (2) have the same characteristic polynomials, then (3) has the solution $Y \sim Z(Y^{(0)} + Y^{(1)} \alpha^{-1} + \cdots)$, where Z is any solution of (2). Suppose that the characteristic polynomials $\mu_i = \sum_0^{r-1} \mu_i^{(\nu)}(x) \alpha^{r-\nu}$ of (2) are such that the numbers $\Re(\mu_i - \lambda_i)$ are bounded above or below and $\mu_i^{(0)} - \lambda_j^{(0)} \not\equiv 0$ or, if $\mu_i^{(0)} - \lambda_j^{(0)} \equiv 0$, then $\mu_i^{(\nu)} - \lambda_j^{(\nu)} \equiv 0$

($\nu=0, \cdots, l_\mu-1$) but $\mu_i^{(l\mu)}-\lambda_i^{(l\mu)}\neq 0$; then (1) has the solution $Y\sim\alpha^{\nu+l-r}Z(Y^{(0)}+Y^{(1)}\alpha^{-1}+\cdots)$.

W. J. Trjitzinsky (Urbana, Ill.).

Bilharz, Herbert. Zur Reduktion von Bewegungsgleichungen auf die Lagrangesche Form. Math. Z. **49**, 583–592 (1944). [MF 11985]

The paper is limited to linear equations of motion and hence is largely a discussion of the concept of self-adjointness as applied to linear systems of differential equations. The main purpose of the paper appears to be twofold: (1) to give an example of a simple system which is self-adjoint in the wider sense of Bliss [Trans. Amer. Math. Soc. **28**, 561–584 (1926), in particular, p. 569] but is not self-adjoint in the usual sense and (2) to remark that definitions of self-adjointness based on the adjoint operator lead to sufficient and necessary conditions for the representation of the given system as Euler equations (of a suitable problem in the calculus of variations) only if a rather unnatural hypothesis of regularity is introduced. This hypothesis, to the effect that the coefficients of the second order terms are twice differentiable, is in practice usually satisfied.

D. C. Lewis (New York, N. Y.).

Parodi, Maurice. Remarque sur la propagation des ondes thermiques sinusoïdales dans un milieu hétérogène. J. Phys. Radium (8) **5**, 23–24 (1944). [MF 12340]

The equations

$$P(x,t)=-K(x)\theta_x(x,t),\quad P_x=-\gamma(x)\theta_t(x,t)$$

are solved formally under the assumption that P and θ are of the form $P=g(x)\exp(iwt)$, $\theta=f(x)\exp(iwt)$. [The author's expression $AD-BC$ is equivalent to the Abel identity for second order differential equations.] *F. G. Dressel.*

Rey Pastor, J. Mixed Dirichlet problems. Revista Union Mat. Argentina **10**, 78–83 (1945). (Spanish) [MF 12277]

The author discusses the boundary-value problem of determining an analytic function $f(z)=u+iv$ in a domain D when the boundary curve C of D is divided into $2n$ arcs and the values of u and v are given on alternate arcs. It is pointed out that the problem can be reduced to a simplified one of determining $f(z)$ when C is divided into two arcs α, β, and the values of one of the conjugate harmonic functions u, v are given on α while the other function vanishes on β; the solution of the general problem is a sum of solutions of simplified problems. The simplified problem is shown to be essentially equivalent to a Dirichlet problem and three methods of solving it are indicated. *E. F. Beckenbach.*

Beckenbach, E. F. Concerning the definition of harmonic functions. Bull. Amer. Math. Soc. **51**, 240–245 (1945). [MF 12260]

The following theorem, comparable to the Looman-Menchoff theorem for analytic functions, is proved. If the real function $u(x, y)$ and its first derivatives with respect to x and y are continuous in the finite domain D; if $\partial u/\partial x$ and $\partial u/\partial y$ are differentiable, or even have finite Dini derivatives, with respect to x and y at all points of D except at most at the points of a denumerable set; and if the Laplace equation,

$$\Delta u=\partial^2 u/\partial x^2+\partial^2 u/\partial y^2=0,$$

is satisfied at almost all points at which $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist in D, then $u(x, y)$ is harmonic in D.

L. H. Loomis (Cambridge, Mass.).

Pólya, G., and Szegö, G. Inequalities for the capacity of a condenser. Amer. J. Math. **67**, 1–32 (1945). [MF 11949]

In this paper the authors obtain a large number of inequalities relating the capacity of a condenser in two or three dimensions to the lengths, areas, volumes, diameters, mean curvature integrals, etc., of the curves, surfaces, and solids involved. The results are too numerous for all of them to be listed in this review; in fact, the first five pages of the paper are devoted to a summary of the various inequalities obtained.

A large number of the results in three dimensions are obtained in a systematic way by using three kinds of symmetrization: the Steiner symmetrization with respect to a plane, the Schwarz symmetrization with respect to a line, and symmetrization with respect to a point. An important theorem is that all three kinds of symmetrization decrease or leave unchanged the capacity of a condenser. A corresponding theorem is proved for the two kinds of symmetrization possible in two dimensions. The proofs are made by regarding the capacity as the minimum value of the Dirichlet integral for suitable boundary values and computing analytically that a symmetrization does not increase the value of that integral. Most of the results obtained in this way are new, but a number of classical theorems are also established incidentally. In the two-dimensional case, for example, some of the symmetrization results reduce to the area theorems of Bieberbach. One important inequality is the following: let σ_0 be a closed convex surface in three dimensions, A its area, and M the integral of its mean curvature; let σ_1 be the exterior parallel surface to σ_0 at distance h. Then the capacity C of the condenser formed by σ_0 and σ_1 satisfies $A/h<4\pi C\leqq M+A/h$.

For any solid S of volume V, area A, integral mean curvature M, and capacity C, it is well known that

$$(\tfrac{3}{4}V/\pi)^{\frac{1}{3}}<(\tfrac{1}{4}A/\pi)^{\frac{1}{2}}<\tfrac{1}{4}M/\pi$$

unless S is a sphere. It is proved in the present paper that

$$(\tfrac{3}{4}V/\pi)^{\frac{1}{3}}<C<\tfrac{1}{4}M/\pi,$$

and that, in the special case of an ellipsoid of axes $2a$, $2b$, $2c$,

$$(\tfrac{3}{4}V/\pi)^{\frac{1}{3}}<\tfrac{1}{3}\{(bc)^{\frac{1}{2}}+(ac)^{\frac{1}{2}}+(ab)^{\frac{1}{2}}\}<C<\tfrac{1}{3}(a+b+c)<\tfrac{1}{4}M/\pi$$

unless $a=b=c$. The proofs are based on representations of the capacity of an ellipsoid in terms of elliptic integrals and Legendre polynomials. If S is a solid of capacity C and R_A the radius of the sphere of the same surface area, then C and R_A are not comparable; that is, for some solids $C>R_A$ and for others $C<R_A$. It is proved that for ellipsoids of revolution C/R_A is greater or less than 1 according as the ellipsoid is prolate or oblate. This result is an extension of the work of A. Russell, who showed that the power series for C/R_A in terms of β, the eccentricity of the revolving ellipse, is $1\pm\frac{3}{640}\beta^6+\cdots$, according as the ellipsoid is prolate or oblate.

The paper closes with some inequalities on the capacity of truncated cylinders, and with certain polygons and polyhedra which minimize capacity. *J. W. Green.*

Szegö, G. On the capacity of a condenser. Bull. Amer. Math. Soc. **51**, 325–350 (1945). [MF 12348]

This is a lecture on the capacity of condensers and point sets, based principally on the results obtained by Pólya and Szegö in the paper reviewed above. The definition of capacity is given, and the relation of capacity to the Green's function and to conformal mapping outlined. The properties

of capacity with respect to symmetrization are described, and some of the inequalities relating capacity to the geometry of the condenser are stated. A considerable portion of the paper is devoted to computing in integral and series form the capacity of special condensers: the ellipsoid, torus, spindle, lens, etc., by means of the appropriate coordinate systems. *J. W. Green* (Aberdeen Proving Ground, Md.).

Deny, Jacques. Sur la convergence des suites de potentiels. C. R. Acad. Sci. Paris **218**, 497–499 (1944). [MF 12120]

It is known that, if a sequence of positive mass distributions $\{\mu_n\}$ converges to a distribution μ, then $U^{\mu}(M) = \lim\inf_{n\to\infty} U^{\mu_n}(M)$, except on a set of exterior capacity zero. Here U^{μ} is the potential of μ. The author proves the following extension. If $\{\mu_n\}$ is a sequence of mass distributions, not necessarily positive, which converges to μ, then, for a suitably chosen subsequence $\{\mu_n'\}$, $\lim U^{\mu_n'}(M) = U^{\mu}(M)$ almost everywhere. Applications are indicated to almost subharmonic functions and to sequences of rational functions. *J. W. Green* (Aberdeen Proving Ground, Md.).

Wittich, H. Ganze Lösungen der Differentialgleichung $\Delta u = e^u$. Math. Z. **49**, 579–582 (1944). [MF 11987]

If $f(\zeta)$ is given, the function $u(x, y)$ is an entire solution of the differential equation

$$\Delta u = f(u) \tag{1}$$

if u satisfies the equation and u is represented by its Taylor series in the two independent variables x and y for all finite values of x and y. It is shown that, if (a) $f(\zeta)$ has a continuous second derivative and is positive and convex and (b) the integral $\int^{\infty}\{\int_0^{\zeta} f(\zeta)d\zeta\}^{-\frac{1}{2}}dx$ is finite, then there is no entire solution of (1). This shows in particular that $\Delta u = e^u$ has no entire solution. An example is given to show that condition (b) cannot be omitted altogether.

A. C. Schaeffer (Stanford University, Calif.).

Pleijel, Åke. Le problème spectral de certaines équations aux dérivées partielles. Ark. Mat. Astr. Fys. **30A**, no. 21, 47 pp. (1944). [MF 12126]

The subject is the differential equation

$$u_{xx} + u_{yy} - q(x, y)u + \lambda k(x, y)u = 0, \tag{1}$$

relative to a region S of the (x, y) plane; λ is a parameter, and $q(x, y)$, $k(x, y)$ are given functions. The cases in which S is bounded, $k(x, y)$ admits a positive lower bound, and u fulfills one of the conditions (2) $\partial u/\partial n = 0$, or (3) $u = 0$, on the boundary of S, are classical. In this paper these restrictions on S and $k(x, y)$ are not made. It is, however, assumed that $q(x, y) \geqq 0$ in S and that, with some constant c, $K_1(u) \leqq cD(u, u)$, where

$$K_1(u) = \int\int_S |k| u^2 dxdy,$$

$$D(u, v) = \int\int_S (u_x v_x + u_y v_y + quv)dxdy.$$

The function space $\mathfrak{D}'$ is defined as that in which $D(u, u)$ and $K_1(u)$ exist, and, insofar as the generalization of (1) and (3) is concerned, in which a condition corresponding to (3) is fulfilled. The closure of $\mathfrak{D}'$ is a Hilbert space $\mathfrak{D}$. If $K(u, v) = \iint_S kuvdxdy$, there exists in $\mathfrak{D}$ a bounded Hermitian operator K such that $K(u, v) = D(Ku, v)$. Under the spectral decomposition of $\mathfrak{D}$ relative to K there corresponds to each interval ω of a μ-axis a subspace $\mathfrak{D}(\omega)$ of $\mathfrak{D}$. It is shown that, whenever ω is at a positive distance from the origin, the elements of $\mathfrak{D}(\omega)$ belong to a subspace of $\mathfrak{D}$ in which the operators $k(x, y)$ and $\partial^2/\partial x^2 + \partial^2/\partial y^2 - q(x, y)$ are defined. Thus a spectrum relative to these operators is obtained. In the case of a point spectrum the existence of a complete set of solutions of (1) is thus shown. The nonhomogeneous problems obtainable from (1), (2) and (1), (3) by replacing the term ku in (1) by kf, where $f(x, y)$ is a given function, are also considered. *R. E. Langer.*

Amerio, Luigi. Teoremi di esistenza per le equazioni lineari del secondo ordine, di tipo ellittico, nei domini illimitati. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 12 pp. (1943) = Ist. Naz. Appl. Calcolo (2) no. **151**. [MF 11524]

The first general theorem of this paper gives sufficient conditions for a linear second-order partial differential equation of elliptic type to possess in an unbounded domain a solution which has a certain type of behavior at infinity; the second general theorem is concerned with the uniqueness of such solutions. The results are applied to several examples in two and three dimensions involving the differential equation

$$\nabla^2 u - \lambda u = 0, \qquad \lambda > 0.$$

W. T. Reid (Evanston, Ill.).

Simonov, N. I. Solution of some boundary problems for elliptical systems of linear equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 259–261 (1944). [MF 12038]

Consider the elliptic system with constant coefficients,

$$\sum_{p,q=1}^{n}\sum_{s=1}^{2} a_{ks}^{pq}\partial^2 u^s/\partial x^p \partial x^q = 0, \qquad k = 1, 2,$$

subject to

$$\sum_{s=1}^{2}(r_s^{\sigma}u^s + R_s^{\sigma}\partial u^s/\partial x^n)|_{x^n=0} = f^{\sigma}(x^1, \cdots, x^{n-1}), \quad \sigma = 1, 2,$$

where r_s^{σ}, R_s^{σ} are real constants and f^{σ} is of class $C^{2(n+1)}$. A solution is required in $x^n > 0$. The treatment is the obvious one of taking the Fourier transform. Nothing in the paper is new. *D. G. Bourgin* (Urbana, Ill.).

Sauer, Ludwig. Parametrixmethode zur Lösung von Randwertproblemen. II. Bildung und Eigenschaften der Parametrix. Math. Ann. **119**, 67–130 (1943). [MF 10095]

Part I [Math. Ann. **118**, 385–440 (1942); these Rev. **6**, 157] contained chapter I and part of chapter II. In chapter I the author developed a theory of integral equations for kernels with singularities of the types possessed by "parametrices" (see below) using the usual method of iterated kernels. The first part of chapter II contained hypotheses concerning differentiability, properties required of Green's functions, and other preliminary work. Part II contains the remainder of chapter II and the whole of chapter III. In a future part the author intends to apply the "parametrix method" of Hilbert to show the existence of solutions of equations of the form

$$EU \equiv AU_{xx} + 2BU_{xy} + CU_{yy} + A_0U_x + B_0U_y + C_0U = F$$

$(AC - B^2 = 1, A > 0)$, which vanish on the boundary d of a region b.

A parametrix is a function $P(t, t')$ $(t = (x, y), t' = (x', y'))$ which satisfies the following conditions. (I) P is of class C''

for $t \neq t'$, $|P| \leqq K_1(1+|\log R|)$, and $|P_x|$, $|P_y|$, $|P_{x'}|$, $|P_{y'}|$, $|EP|$, $|\bar{E}'P| \leqq K_2/R$ ($R^2=(x'-x)^2+(y'-y)^2$), where $\bar{E}$ is the adjoint differential operator of E. (II) P has a singularity for $R \to 0$ such that

$$\int_b P \cdot (E'U')dt' = \int_b (\bar{E}'P) \cdot U' dt' - U,$$

$$E\left(\int_b PF'dt'\right) = \int_b (EP) \cdot F' dt' - F$$

($F'=F(t')$, etc.); (III) $\int_b PF'dt'$ has the same differentiability properties as the functions U. In other words, P has all the properties of a Green's function except for the requirements that $EG=\bar{E}'G=0$ for $R>0$. Any solution of the differential equation becomes, by virtue of (II), a solution of the integral equation

$$U - \int_b (E'P)U'dt' = -\int_b PF'dt'.$$

The remainder of chapter II is concerned with the explicit representation of parametrices (1) for the general equation for a region b with a boundary d which has a continuously turning inner normal (with certain further differentiability properties) but may have double points, and (2) for the equation with $A=C=1$, $B=0$ in which d may have a finite number of corners whose angles are of the forms π/g_i, where g_i is an integer greater than 1. Chapter III is concerned with properties of P such as bounds for derivatives, properties of iterated kernels, etc. *C. B. Morrey, Jr.*

Vranceanu, G. Geometrische Eigenschaften der partiellen linearen Differentialgleichung zweiter Ordnung. Mathematica, Timişoara **20**, 98–112 (1944). [MF 12455]

The partial differential equation of second order,

$$(1) \qquad a^{rs}\frac{\partial^2 z}{\partial x^r \partial x^s} + a^r \frac{\partial z}{\partial x^r} + az = 0, \quad r, s = 1, \cdots, n,$$

is studied under the pseudogroup of transformations $z'=\alpha(x^1, \cdots, x^n)z$, $x'^r=x'^r(x^1, \cdots, x^n)$. In the case when $|a^{rs}| \neq 0$, it is proved that the invariants of (1) are the simultaneous invariants of an equation of Monge of the second degree and an exterior quadratic differential form of the second degree. When $n=2$, a Riemannian metric can be intrinsically introduced if a certain relative invariant does not vanish. The problem is then studied by means of the method of orthogonal congruences of Ricci (method of moving frames of Élie Cartan), which allows a far-reaching normalization of the equations; the normalization amounts geometrically to taking the principal directions of the Monge cone as directions of the orthogonal congruences and reducing the exterior differential form to a normal form under orthogonal transformations. The problem of equivalence is solved on the basis of this reduction. The parabolic case $|a^{rs}|=0$ is also discussed. *S. Chern* (Princeton, N. J.).

Petrowsky, I. Sur la diffusion des ondes et les lacunes pour les systèmes d'équations hyperboliques. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 101–106 (1944). (Russian. French summary) [MF 11461]

This is a short expository lecture on the diffusion problem in the theory of linear hyperbolic equations. A wave propagation, due to an initial disturbance which is limited in time and space, is said to be without diffusion when the corresponding wave has a sharp front and tail. Hadamard formulated a famous conjecture stating that hyperbolic equations without diffusion are exceptional. The author gives a review of his own work and of Mathisson's solution of the problem [Acta Math. **74**, 249–282 (1939); these Rev. **1**, 120]. The author was evidently unaware of the paper of Hadamard [Ann. of Math. (2) **43**, 510–522 (1942); these Rev. **4**, 45]. *A. Weinstein* (Toronto, Ont.).

Drăganu, Mircea. Sur une solution opératoire de l'équation de la propagation des ondes. Mathematica, Timişoara **20**, 69–72 (1944). [MF 12471]

The author derives the Cauchy solution for the partial differential equation

$$(*) \qquad a^{-2}u_{tt} + 2bu_t = \nabla^2 u + 4\pi\omega(x, y, z, t),$$

that is, the solution with u and u_t assigned for $t=0$. This is accomplished by applying the unilateral Laplace transform to (*) with respect to t and solving the resulting spatial equation by classical techniques. *A. E. Heins.*

Sneddon, Ian N. The symmetrical vibrations of a thin elastic plate. Proc. Cambridge Philos. Soc. **41**, 27–43 (1945). [MF 12373]

If w is the normal displacement of the plate, r the radial coordinate, t the time, Z the load, ρ the density of the plate material, h the thickness and $b^2=D/(2\rho h)$, where D is the flexural rigidity, then for the axially symmetrical case the differential equation of the plate is

$$(1) \qquad b^2(\partial^2/\partial r^2 + r^{-1}\partial/\partial r)^2 w + \partial^2 w/\partial t^2 = Z(r, t)/(2\rho h).$$

The usual operational or Laplace transform method is to eliminate the time derivative and solve the resultant ordinary differential equation in r. The author eliminates the r-derivative for an infinite plate by using the Hankel transform. If

$$\bar{w}(\xi; t) = \int_0^\infty rw(r; t)J_0(\xi r)dr, \quad \bar{Z}(\xi; t) = \int_0^\infty rZ(r; t)J_0(\xi r)dr,$$

(1) is replaced by

$$d^2\bar{w}/dt^2 + b^2\xi^4\bar{w} = \bar{Z}(\xi; t)/(2\rho h).$$

The solution $\bar{w}$ is determined by proper initial conditions. Then w is obtained by the inversion formula

$$w(r; t) = \int_0^\infty \xi \bar{w}(\xi; t)J_0(\xi r)d\xi.$$

A similar method is used for plates of finite radius. The advantage of the present method is its directness in obtaining the result under initial displacement or applied external forces. The general result obtained is reduced to that of Boussinesq and Zener for force concentrated at the center of the plate. *H. S. Tsien* (Pasadena, Calif.).

Erouguine, N. Sur les solutions fonctionnellement invariantes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 371–372 (1944). [MF 11642]

Consider $\nabla^2 u = u_{tt}$. A functionally invariant solution is one such that $F(u)$ is a solution for arbitrary choice of F. The most general solutions of this sort are implicitly determined by $x+f(u)y+g(u)z+(1+f^2(u)+g^2(u))t+h(u)=0$,

where f, g, h are arbitrary. A subset of such solutions is given by $u=\phi(x+iz(1+c^2)^{\frac{1}{2}}+ict;\ y+cz+t(1+c^2)^{\frac{1}{2}})$.

D. G. Bourgin (Urbana, Ill.).

Gogoladze, V. G. The Fourier integral and functionally invariant solutions of the wave equation in n-dimensional space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 307–310 (1944). [MF 12040]

Consider the two-dimensional wave equations of elasticity theory in the potentials ϕ and ψ: $a^2\nabla^2\phi=\varphi_{tt}$, $b^2\nabla^2\psi=\psi_{tt}$. The problem solved amounts to the determination of ϕ (or ψ) corresponding to a force acting normally on a segment X of the x-axis for $t>0$. The solution may be written down at once by the Cauchy method as an integral of Fourier type. This is essentially what the author does. The major part of the paper is taken up with a reduction of this fourfold integral to a threefold integral by using contour integration with respect to one of the parameters, following the method (choice of integration variables and use of $\int^x\phi dy$) associated with the name of Zeilon in modern applied mathematics texts. The final formula seems interesting and natural. *D. G. Bourgin* (Urbana, Ill.).

Olevsky, M. Solution du problème de Cauchy pour l'équation des ondes dans un espace à n dimensions à courbure constante. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 3–6 (1945). [MF 12544]

Let Δ_2 represent the second differential parameter of Beltrami for a space $R_n (n>2)$ of constant curvature k. The solution of the following problem is given:

$$\Delta_2 u=\partial^2 u/\partial t^2,\quad u(P,t)\,|_{t=0}=f(P),\quad \partial u/\partial t\,|_{t=0}=\varphi(P),$$

where $P\equiv(x_1,\cdots,x_n)$ is a point in the space R_n. The problem was previously stated by the author [same C. R. (N.S.) **33**, 282–287 (1941), where explicit formulas for the solution of the problem were given only for $n=3$; see these Rev. **5**, 98 for the methods used]. *F. G. Dressel.*

Olevsky, M. Quelques théorèmes de la moyenne dans les espaces à courbure constante. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 95–98 (1944). [MF 12573]

[Cf. the preceding review.] Let $R_n (n\geqq 2)$ be a space of constant curvature k, and $P\equiv(x_1,\cdots,x_n)$ a point in R_n. The mean value $V(u,P,s)$ of a function $u(P)$ is defined by the integral

$$V(u,P,s)=w^{-1}\int_{F_s} u\,dw,\qquad w=\int_{F_s} dw,$$

where F_s is the surface of a "sphere" with center at P and geodesic radius s. The author shows that the mean value of u satisfies the equation

$$\partial^2 V/\partial s^2+(n-1)k^{\frac{1}{2}}\cot(sk^{\frac{1}{2}})\partial V/\partial s=\Delta_2 V,$$

where Δ_2 is the second differential parameter of Beltrami for the space R_n. The author writes a formula for V in terms of $u, \Delta_2 u, \Delta_2(\Delta_2 u), \cdots$, which for $k\to 0$ reduces to the formula of Pizzetti for the mean value of a function in a Euclidean space. The following generalization of a theorem of Åsgeirsson is also stated. Let u be a function of two points P and $Q\equiv(y_1,\cdots,y_n)$ of R_n, which satisfies $\Delta_2 u=\Delta_2 u$; then

$$\int_{F_s(P)} u(\xi,Q)\,d\omega_\xi=\int_{F_s(Q)} u(P,\eta)\,d\omega_\eta.$$

F. G. Dressel (Durham, N. C.).

Lowan, Arnold N. On the problem of heat conduction in a semi-infinite radiating wire. Quart. Appl. Math. **3**, 84–87 (1945). [MF 12124]

A formula is derived for the temperatures in a slender semi-infinite wire whose surface is subject to linear heat transfer into a medium at constant temperature. Arbitrary initial and end temperatures are assumed.

R. V. Churchill (Ann Arbor, Mich.).

Lowan, Arnold N. On the problem of heat conduction in thin plates. J. Math. Phys. Mass. Inst. Tech. **24**, 22–29 (1945). [MF 12313]

Six boundary value problems are solved for the temperatures in thin plates with linear heat transfer from their faces into media at constant temperature. The problems consist of two on circular plates in which the flow is radial, one on a plate extending over the entire xy-plane, one for a half-plane and one for a quadrant. Laplace transforms and Green's functions are employed. *R. V. Churchill.*

Jaeger, J. C. Conduction of heat in a slab in contact with well-stirred fluid. Proc. Cambridge Philos. Soc. **41**, 43–49 (1945). [MF 12374]

One face of a finite slab is either insulated or kept at temperature zero and the other is exposed to a fluid whose temperature is assumed uniform at each instant. The flux of heat across the latter surface is proportional to the temperature difference of the surface and the fluid. The formulas for the temperatures in the slab and the fluid are derived first when heat is supplied to the fluid at a constant rate, both slab and fluid being initially at zero temperature; second, when no heat is supplied but the fluid and solid are initially at different uniform temperatures. The formulas are derived by using Laplace transforms. The variation of some of these temperatures with the time is shown graphically and a comparison is made with an approximate solution used in connection with the heating of rooms.

R. V. Churchill (Ann Arbor, Mich.).

Kormes, J. P. The solution of the differential equation $(a^2\partial^2/\partial t^2-\Delta)(\partial^2/\partial t^2-\Delta)u=f(x,y,z,t)$ by Hadamard's method. Bull. Amer. Math. Soc. **50**, 842–855 (1944). [MF 11553]

The equation indicated in the title is considered subject to the initial conditions $u, u_t, u_{tt}, u_{ttt}=0$ for $t=0$. The characteristic conoid is given by

$$\Gamma\Gamma'=[(\tau-t)^2-r^2][(\tau-t)^2/a^2-r^2]=0.$$

The Hadamard method is extended using the elementary solution

$$(2r)^{-1}\{\log[(\tau-t)+r][(\tau-t)/a-r]^a - \log[(\tau-t)-r][(\tau-t)/a+r]^a\}.$$

[There is a somewhat confusing usage of G' for a concept different from $\lim G'_{\epsilon_1,\epsilon_2,\epsilon}$.]

The reviewer offers the following comments. The solution found by the author may be obtained alternatively with a few minutes' work; of course, the main interest of this paper is in the method followed. Next, the problem treated is the simplest case of the degenerate hyperbolic equation of order four. It is not typical, for in the general situation the sections $\tau-t=C$ of the conoids in the product intersect. When the conoids do not intersect, except for $C=0$, the Hadamard scheme extends in detail as one would expect and as the paper verifies competently. *D. G. Bourgin.*

Charazov, D. F. A general representation of the solutions of elliptic differential equations of higher than second order, in multiply connected domains. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 799–806 (1941). (Russian. Georgian summary) [MF 10303]

The author gives a formula for the representation in multiply connected domains of solutions of certain linear partial differential equations of elliptic type of order $2n$, $n \geqq 1$. He represents the solutions in terms of n analytic functions of a complex variable. [Some statements of the author seem to require a more detailed and exact formulation.] *S. Bergman* (Providence, R. I.).

Trjitzinsky, W. J. Analytic theory of parametric linear partial differential equations. Rec. Math. [Mat. Sbornik] N.S. 15 (57), 179–242 (1944). (English. Russian summary) [MF 12286]

The paper deals with equations of order m in the n variables $x_1, \cdots, x_n$. The coefficients, besides being functions of the variables, differentiable with respect to them as may be needed, depend also upon a parameter λ. They are assumed to be analytic functions of λ, each asymptotic to the product of some power of λ by a formal power series in $1/\lambda$. Formal series solutions of types given in simpler cases by G. D. Birkhoff are considered. Depending upon the structure of a certain characteristic equation, these series are appropriately of a "normal" or of an "anormal" type.

The discussion is subsequently turned to the consideration of actual solutions that are asymptotic to the formal series. In this the restriction $m=2$ applies, and λ is largely restricted to be real. The elliptic and the normal hyperbolic cases are dealt with. The results are summarized in appropriate existence theorems. These generally apply to regions of the variable and parameter that are subject to a number of restrictive and delimiting hypotheses which are dictated by the method and admit of no concise summaries. In the words of the author, "such regions may or may not exist, depending upon the particular problem on hand; they will exist, of course, in an extensive variety of cases."

R. E. Langer (Madison, Wis.).

Theory of Probability

Bartlett, M. S. Negative probability. Proc. Cambridge Philos. Soc. 41, 71–73 (1945). [MF 12379]

Negative probabilities are introduced starting from the formal equation $p=p_1+p_2$. "If p denotes the chance of a biased penny giving 'heads,' and for p_1 we write $\frac{1}{2}$, p_2 is a correcting term." For $p<\frac{1}{2}$ we have "negative probabilities." The author dwells on the fact that they cannot be interpreted as limits of frequencies, and calls a random variable assuming values with negative probabilities an "extraordinary" random variable. "It has been shown that orthodox probability theory may consistently be extended to include probability numbers outside of the conventional range." *W. Feller* (Ithaca, N. Y.).

Reboul, G., et Reboul, J.-A. Probabilités mathématiques et probabilités physiques. Applications. J. Phys. Radium (8) 5, 108–116 (1944). [MF 12333]

In addition to mathematical probability the authors introduce a physical probability essentially as follows. "Si l'on produit une variation dU de l'actif U d'un certain facteur réparti sur les N individus constituant le système, cette variation touchera dN d'entre eux. La probabilité mathématique . . . est dN/N, . . . le rapport dU/U constitue la probabilité physique." A number of physical laws are derived as an immediate consequence of the principle that the sum of the probabilities of changes corresponding to a variation of conditions of a system is zero. [It seems to the reviewer that the method can be used to derive any physical law, real or imaginary.] *W. Feller.*

Chung, Kai-Lai, and Hsu, Lietz C. A combinatorial formula and its application to the theory of probability of arbitrary events. Ann. Math. Statistics 16, 91–95 (1945). [MF 12362]

The author generalizes an inversion formula of formal logic by introducing a generalized Möbius function. He applies his results to the theory of probability. In particular he obtains a simple proof of a generalization of Poincaré's formula. *A. H. Copeland* (Ann Arbor, Mich.).

Fan, Ky. Un théorème général sur les probabilités associées à un système d'événements dépendants. C. R. Acad. Sci. Paris 218, 380–382 (1944). [MF 12110]

The author observes that most known equalities (or inequalities) relating probabilities of combinations of events are reducible to a form

$$(*) \qquad \sum\{a_k S_k + b_k P_{[k]}\} = 0, \geqq 0,$$

where S_k and $P_{[k]}$ are the probabilities of the occurrence of at least and exactly k events, respectively, and the a_k, b_k are constants; these relations are more easily proved for independent events. He shows that, if such a relation is true for independent events, then it is automatically true for dependent events. The proof consists in replacing the probabilities of the independent events by 1's and 0's, which represent the successes and failures of the corresponding dependent events, and then taking the expected value of the result. *A. H. Copeland* (Ann Arbor, Mich.).

Loève, M. Systèmes d'événements en nombre fini. Ann. Univ. Lyon. Sect. A. (3) 5, 55–74 (1942). [MF 12368]

The topics covered in this paper are roughly the same as those in Fréchet's book, Les probabilités associées à un système d'événements compatibles et dépendants, part I [Actualités Sci. Ind., no. 859, Hermann, Paris, 1940; these Rev. 3, 168]. Some of the identities and inequalities are given in generalized form and there is greater emphasis on symbolic methods. *I. Kaplansky* (New York, N. Y.).

Fréchet, Maurice. Les systèmes d'événements et le jeu des rencontres. Revista Mat. Hisp.-Amer. (4) 4, 95–126 (1944). [MF 12165]

This paper consists of extracts, largely without proofs, from the author's book, Les probabilités associées à un système d'événements compatibles et dépendants, part II, Cas particuliers et applications [Actualités Sci. Ind., no. 942, Hermann, Paris, 1943], which was not available to the reviewer. The contents of the paper are as follows. (1) Inequalities on the mean, mode and median of a positive random variable; for example, the median is at most twice the mean. (2) A formula for the variance of runs in samples from a binomial population. (3) Symmetric systems of events (where the joint probability p_k of k events depends only on k). Conditions are given for the existence of events with prescribed p's. (4) The "problème des rencontres."

A generalization given by the reviewer [Amer. Math. Monthly 46, 159–161 (1939)] is amplified by giving several alternative symbolic forms and expressions in terms of Laguerre polynomials. The reader should be alert for a substantial number of typographical errors.

I. Kaplansky (New York, N. Y.).

Linés Escardó, E. Problems of coincidence. II. Revista Mat. Hisp.-Amer. (4) 4, 188–205 (1944). (Spanish) [MF 12173]

[Part I appeared in the same Revista (4) 1, 292–314 (1942).] Suppose that two urns contain balls, some bearing the number 1, some the number 2, etc. Drawings are made from both urns and a coincidence is recorded when the numbers match. Required: the probability of exactly r coincidences. The author distinguishes two problems: in the first, balls are replaced after drawing in one of the urns; in the second, balls are replaced in neither urn. The first problem is ascribed to the reviewer and Fréchet, and the author then proceeds to solve the second. There appears to be a fundamental misunderstanding, inasmuch as the first problem is trivial, while the second is the generalized "problème des rencontres" for which the reviewer and Fréchet have given symbolic solutions [cf. the preceding review]. The author's solution is direct but leads to cumbersome formulas. *I. Kaplansky* (New York, N. Y.).

Mullemeister, Hermance. Mean lengths of line segments. Amer. Math. Monthly 52, 250–252 (1945). [MF 12443]

If a unit segment is divided at random into two parts, then the mean length of the longer segment is $\frac{3}{4}$ and that of the shorter segment is $\frac{1}{4}$, provided the probability distribution is uniform. The author considers the general case in which the segment is divided into $n+1$ parts arranged in descending order of magnitude. The computation is simplified by noting that mean values are related to centroids.

A. H. Copeland (Ann Arbor, Mich.).

Greville, T. N. E. On multiple matching with one variable deck. Ann. Math. Statistics 15, 432–434 (1944). [MF 11763]

The author obtains compact forms for the mean and variance of the number of correct guesses in a multiple matching problem. He makes use of a counting function introduced by Wilks and improved by Battin.

A. H. Copeland (Ann Arbor, Mich.).

Wilkinson, Roger I. The combination of probability curves in engineering. Trans. Amer. Inst. Elec. Engrs. 61, 953–963 (1942). [MF 12580]

Fan, Ky. Entfernung zweier zufälligen Grössen und die Konvergenz nach Wahrscheinlichkeit. Math. Z. 49, 681–683 (1944). [MF 11981]

The distance (X, Y) of the two random variables X, Y is defined as the greatest lower bound of all positive ϵ for which $\Pr\{|X-Y|\geqq\epsilon\}<\epsilon$. With this definition the space of random variables is complete and metric convergence is the same as convergence in probability. *W. Feller.*

Kalecki, M. On the Gibrat distribution. Econometrica 13, 161–170 (1945). [MF 12339]

The author discusses some assumptions from which one could conclude that the logarithms of certain economic variates (like income) are normally distributed. The variates are treated as random variables subject to changes in time. The assumptions are quite formal and are stated only in an incomplete manner. *W. Feller* (Ithaca, N. Y.).

Orts, J. M.[a] Convergence of some mean values. Revista Mat. Hisp.-Amer. (4) 4, 127–130 (1944). (Spanish) [MF 12166]

Let r and s be fixed positive integers, $0\leqq s<r$. Let $0<p<1$. The author proves that

$$\lim_{n\to\infty}\sum\binom{n}{m+s}p^{m+s}(1-p)^{n-m-s}=1/r.$$

W. Feller (Ithaca, N. Y.).

Stephan, Frederick F. The expected value and variance of the reciprocal and other negative powers of a positive Bernoullian variate. Ann. Math. Statistics 16, 50–61 (1945). [MF 12357]

The author obtains series, in forms suitable for computation, for $E(X^{-a})$, where X is a positive Bernoullian variate: that is, Prob $\{X=k\}=\binom{n}{k}p^kq^{n-k}/(1-q^n)$ for $k=1, \cdots, n$. Similar results are obtained for Bernoullian variates with several excluded values, and for hypergeometric variates.

D. Blackwell (Washington, D. C.).

Tweedie, M. C. K. Inverse statistical variates. Nature 155, 453 (1945). [MF 12364]

In a sequence of Bernoulli trials let Y be the number of trials up to the rth success. The random variable Y is called the inverse of the ordinary Bernoulli variate X. The author finds the seminvariant-generating functions of X and Y and points out that they satisfy a relation which occurs also in the theory of Brownian motion. A detailed exposition is to follow. *W. Feller* (Ithaca, N. Y.).

Mays, W. J. The valuation of risks. Amer. Math. Monthly 52, 138–148 (1945). [MF 12064]

D. Bernoulli's "moral expectation" is generalized by introducing a continuous function f of one variable for the valuation of the risk. Let p_i be the probability of a gain (or loss) P_i $(i=1, \cdots, n;\ \sum p_i=1)$. The moral expectation X is then defined by the equation

$$f(A+X)=p_1f(A+P_1)+\cdots+p_nf(A+P_n).$$

Here $f(A+P)$ denotes the valuation which a gain (or loss) P has for a person whose fortune is A. This definition reduces to the definition of mathematical expectation if $f(x)=x$ and to the definition of Bernoulli's moral expectation if $f(x)=\log x$. The author also considers the case $f(x)=x^m$ with $0<|m|\leqq 1$ and discusses the Petersburg problem and the problem of the ruin of gamblers in repeated plays. *E. Lukacs* (Berea, Ky.).

Esseen, Carl-Gustav. On the Liapounoff limit of error in the theory of probability. Ark. Mat. Astr. Fys. 28A, no. 9, 19 pp. (1942). [MF 12327]

Let $\{X_k\}$ be mutually independent random variables with the same distribution function $F(x)$. Suppose that the first moment of $F(x)$ vanishes and that the second moment σ^2, the third moment α_3 and the third absolute moment β_3 are finite. Let

$$S_n=\{X_1+\cdots+X_n\}/(\sigma n^{\frac{1}{2}})$$

and denote by $F_n(x)$ the distribution function of S_n, by $\Phi(x)$ the normalized Gaussian distribution. Improving on well-known results of Liapounoff and Cramér, the author proves that there exists a constant C such that

$$(1)\qquad |F_n(x)-\Phi(x)|\leqq C\beta_3\sigma^{-3}n^{-\frac{1}{2}}.$$

The same result was established independently by A. C. Berry [Trans. Amer. Math. Soc. 49, 122–136 (1941); these Rev. 2, 228]. Berry's method, which is somewhat simpler, was recently improved by P. L. Hsu [cf. the second review below].

The author shows that, if the characteristic function $f(t)$ of $F(x)$ is not periodic, the asymptotic expansion

$$(2)\quad F_n(x)=\Phi(x)+\tfrac{1}{6}\alpha_3\sigma^{-3}(2\pi n)^{-\frac{1}{2}}(1-x^2)e^{-x^2/2}+o(n^{-\frac{1}{2}})$$

holds. It is stated that most of the results and methods of the paper can be generalized to the case where the X_k have different distribution functions. *W. Feller.*

Esseen, Carl-Gustav. Determination of the maximum deviation from the Gaussian law. Ark. Mat. Astr. Fys. **29A**, no. 20, 10 pp. (1943). [MF 12020]

Continuing the investigations reviewed above, the author derives an asymptotic expansion analogous to (2) and valid in the case where $f(t)$ is periodic. The second term on the right is then replaced by a more complicated expression involving the function $[x]-x+\frac{1}{2}$. *W. Feller.*

Hsu, P. L. The approximate distributions of the mean and variance of a sample of independent variables. Ann. Math. Statistics **16**, 1–29 (1945). [MF 12355]

The notations are the same as in the second review above. The author first gives a new and greatly simplified derivation of Cramér's well-known asymptotic expansion for $F_n(x)-\Phi(x)$. The simplification results from the use of Cesàro type kernels instead of the Riesz kernels used by Cramér; these were first introduced by A. C. Berry to deduce formula (1); a simple self-contained proof of (1) is also given in the present paper.

Suppose now that $\sigma=1$ and that the fourth moment α_4 exists. Let $U=n^{-1}\{\sum_1^n X_k^2-S_n^2\}$, and let $G(x)$ be the distribution function of $n^{\frac{1}{2}}(U-1)(\alpha_4-1)^{-\frac{1}{2}}$. Using the same technique, the author derives asymptotic expansions for $G(x)$ which are analogous to Cramér's expansions.

W. Feller (Ithaca, N. Y.).

Bobroff, A. A. Conditions of applicability of the strong law of large numbers. Duke Math. J. **12**, 43–46 (1945). [MF 12069]

Kolmogoroff's well-known sufficient condition for the strong law of large numbers is proved in a way which is, as the author points out, substantially a reproduction of the method of Kac [Studia Math. **6**, 46–58 (1936)]. Next, in a search for necessary conditions, the author gives a slight generalization of the condition of Halmos [Ann. of Math. (2) **40**, 800–804 (1939); these Rev. **1**, 62]. Actually the author's condition is clearly necessary even for the weak law and can easily be formulated in stronger forms.

W. Feller (Ithaca, N. Y.).

Kimball, Bradford F. Note on asymptotic value of probability distribution of sum of random variables which are greater than a set of arbitrarily chosen numbers. Ann. Math. Statistics **15**, 423–427 (1944). [MF 11760]

Denoting by $I(u)$ the $(n-1)$-fold integral

$$\int\cdots\int \exp\left[-k\left(\sum_{i=1}^{n} x_i^2\right)dx_1\cdots dx_{n-1}\right]$$

over the region $\{x_1+\cdots+x_n=u,\ x_i\geqq a_i\}$, where $a_1,\cdots,a_n$ are constants, $a_1+\cdots+a_n=u_0<u$, $k>0$, the author shows that

$$\lim_{u\to\infty} n^{\frac{1}{2}}e^{ku^2/n}I(u)=(\pi/k)^{(n-1)/2}.$$

This implies that, if $x_1,\cdots,x_n$ are independent normal chance variables with mean zero and variance unity, for every $\delta>0$ we can find u' such that, for $u\geqq u'$,

$$(1-\delta)p_A(u)\leqq p_B(u)P\{x_i\geqq a_i,\ i=1,\cdots,n\}\leqq p_A(u),$$

where

$$p_A(u)du=\text{Prob}\{u-\tfrac{1}{2}du\leqq x_1+\cdots+x_n\leqq u+\tfrac{1}{2}du\}=P(E)$$

and $p_B(u)$ is the conditional probability of E under the hypothesis $\{x_i\geqq a_i,\ i=1,\cdots,n\}$. *D. Blackwell.*

Orts, J. M.[a] On certain iterated probabilities. Revista Mat. Hisp.-Amer. (4) **4**, 153–158 (1944). (Spanish) [MF 12170]

Let $a_1,\cdots,a_N$ be positive real numbers with $a_1+\cdots+a_N=1$. Let Δ be the circulant matrix whose rows are the cyclic permutations of $a_1,\cdots,a_N$. It is shown that $\|\Delta\|<1$ and that Δ^n converges. The problem is, as the author states, a simple special case of a Markov chain. *W. Feller.*

Cernuschi, Felix, and Saleme, Ernesto. A new scheme of contagion in probability. An. Soc. Ci. Argentina **138**, 201–213 (1944). (Spanish) [MF 12138]

Translated into the customary language of Markov chains the problem can be restated as follows. There are only two states E_1, E_2 (success and failure). The matrix of transition probabilities is constant. Required the probability of k successes in n trials. The authors obtain the generating function for this probability. *W. Feller* (Ithaca, N. Y.).

Blackwell, David. The existence of anormal chains. Bull. Amer. Math. Soc. **51**, 465–468 (1945). [MF 12530]

Doblin, in making a detailed analysis of Markov processes [Ann. Sci. École Norm. Sup. (3) **57**, 61–111 (1940); these Rev. **3**, 3], defined a normal chain (essentially the nondegenerate case) in a way too detailed to reproduce here. The author finds a simple condition that a chain be normal and gives an elementary example of an anormal chain.

J. L. Doob (Washington, D. C.).

Kac, M. Random walk in the presence of absorbing barriers. Ann. Math. Statistics **16**, 62–67 (1945). [MF 12358]

Let $X_1, X_2,\cdots$ be independent symmetric random variables and denote by N the smallest n for which either $X_1+\cdots+X_n>p$ or $X_1+\cdots+X_n<-q$ $(p, q\geqq 0)$. The random variable N represents the "length of life" of a particle starting at the origin at time zero and receiving a displacement X_i at time i. In the special case where $X_i=\pm 1$ with probability $\frac{1}{2}$, an exact formula for $P\{N>n\}$ is given, which implies a result of Furth in the theory of Brownian motion. When X_i has a continuous density function $\rho(x)$,

$$P\{N>n\}=\sum_{j=1}^{\infty}\lambda_j^n f_j(q)\int_0^{p+q} f_j(t)dt,$$

where $\lambda_1, \lambda_2,\cdots$ are the eigenvalues and $f_1, f_2,\cdots$ the eigenfunctions of the integral equation

$$\int_0^{p+q}\rho(s-t)f(t)dt=\lambda f(s).$$

D. Blackwell (Washington, D. C.).

Rice, S. O. Mathematical analysis of random noise. Bell System Tech. J. **24**, 46–156 (1945). [MF 11953]

This is the second half of a paper whose first half was reviewed previously [same J. **23**, 282–332 (1944); these

Rev. 6, 89]. Starting with the representation

$$I(t)=T^{-\frac{1}{2}}\sum_{k=1}^{N}B(\omega_k)(X_k\cos\omega_k t+Y_k\sin\omega_k t),\qquad \omega_k=2\pi k/T,$$

for the noise current, the author calculates the average number per second of zeros of $I(t)$ and of maxima and minima. For large N and T the average number of zeros is given by the simple formula

$$\pi^{-1}\{-\psi''(0)/\psi(0)\}^{\frac{1}{2}},$$

where

$$\psi(t)=\int_0^\infty B^2(f)\cos 2\pi ft\,df$$

is the "correlation function" of $I(t)$, and $\psi''(0)$ is assumed to exist. The author also calculates the probability that a zero of $I(t)$ should lie in (t_1, t_1+dt_1) and another zero in (t_2, t_2+dt_2). [All these results can also be derived using the methods introduced by the reviewer [Bull. Amer. Math. Soc. **49**, 314–320, 938 (1943); Amer. J. Math. **65**, 609–615 (1943); these Rev. **4**, 7; **5**, 96, 179].] Agreement between these theoretical results and experimental data obtained by M. E. Campbell is discussed. A hitherto unsolved problem of great practical importance is to determine the probability that no zero of $I(t)$ should fall within a prescribed interval. The author gives an expression for this probability in the form of an infinite series but this is inadequate for computational purposes. Several sections are devoted to the investigation of analogous problems for the so-called "envelope" of $I(t)$ which is obtained experimentally by passing the noise current through a rectifier.

The author studies statistical properties of the random variable $E=\int_0^\tau I^2(t)dt$, which is proportional to the noise energy. It is shown by an intuitive argument that the distribution of E is almost normal for large τ, and an approximate expression is proposed for the distribution of E when τ is small or moderate. He discusses statistical properties of the superposition of noise $I(t)$ and a sinusoidal signal $s(t)=P\cos(\omega t-\varphi)$.

The last part of the paper is more special and technical in character. It is concerned with the statistical properties of noise and noise plus signal after they have passed through various nonlinear devices (linear or square law detectors, etc.). The author presents here not only his own contributions but also those of Bennett, Middleton, North and Van Vleck. *M. Kac* (Ithaca, N. Y.).

Bishop, D. J. The renewal of aircraft. Ministry of Aircraft Production, Aeronaut. Res. Committee, Rep. and Memoranda no. 1907 (6342), 12 pp. (1942). [MF 11960]

Practical (in part numerical) applications of the standard methods of renewal theory. *W. Feller* (Ithaca, N. Y.).

Mathematical Statistics

Gonin, H. T. Curve fitting by means of the orthogonal polynomials in binomial statistical distributions. Trans. Roy. Soc. South Africa **30**, 207–215 (1944). [MF 12154]

This paper deals with the fitting by least squares of the linear combination

$$Y_x=[a_0G_0(x)+a_1G_1(x)+\cdots+a_rG_r(x)]\phi(x)$$

to a series of data θ_x, having weights $1/\phi(x)$, where $x=0, 1, \cdots, n$. Here

$$\phi(x)=(n!/x!(n-x)!)p^x(1-p)^{n-x},$$

where p is some number between 0 and 1, and the G's are defined in such a way that $\sum_{x=0}^{n}G_r(x)G_s(x)\phi(x)$ is equal to 0 if $r\neq s$ and to $r!n!p^r(1-p)^r/(n-r)!$ if $s=r>0$. The author gives numerical examples. The paper contains numerous misprints. *T. E. Sterne* (Aberdeen Proving Ground, Md.).

Rodrigues, Milton da Silva. On an extension of the concept of moment with applications to measures of variability, general similarity, and overlapping. Ann. Math. Statistics **16**, 74–84 (1945). [MF 12360]

Definitions are given for finite univariate populations and for pairs of such populations. *H. Scheffé.*

Morrell, A. J. H. Note on Wilson and Hilferty's approximation to the χ^2-distribution. J. Roy. Statist. Soc. (N.S.) **107**, 59 (1944). [MF 11964]

It is shown that the $\chi^{\frac{1}{3}}$-distribution, used for the Wilson and Hilferty approximation, is almost symmetrical.

R. L. Anderson (Princeton, N. J.).

Pierce, Joseph A. On the summation of progressions useful in time series analysis. J. Amer. Statist. Assoc. **39**, 387–389 (1944). [MF 12199]

Niessen, A. M. On the summation of certain progressions useful in time series analysis. J. Amer. Statist. Assoc. **40**, 98–100 (1945). [MF 12200]

Descriptions of ways of computing

$$a^r+(a+1)^r+\cdots+(a+n)^r,$$

not using the known standard formulas. *W. Feller.*

Hsu, L. C. Some combinatorial formulas with applications to probable values of a polynomial-product and to differences of zero. Ann. Math. Statistics **15**, 399–413 (1944). [MF 11758]

Let x be a discrete random variable assuming the values $0, \cdots, k$, each with the probability $1/k$, and let $(x_1, \cdots, x_n)$ be a sample of x. Let $f_1(x), \cdots, f_n(x)$ be n given polynomials. What is the expectation of the product $f_1(x_1)\cdots f_n(x_n)$ under a condition $x_1+\cdots+x_n=m$, or under this condition and the additional condition $a\leqq x_i\leqq b$? Explicit formulas answering those questions are derived, and corollaries for continuous random variables are presented. For those corollaries, however, the assumptions are not clearly stated, and the proofs of some of them, for example, corollary 3, do not seem conclusive. *Z. W. Birnbaum.*

Scheffé, Henry. A note on the Behrens-Fisher problem. Ann. Math. Statistics **15**, 430–432 (1944). [MF 11762]

The author shows that there exists no symmetric solution to the Behrens-Fisher problem employing the t-distribution. Specifically, if $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ are random samples from two different normal distributions, inferences about the difference δ between the population means can conveniently be made if there exists a quotient $t=(L-\delta)/(Q/k)^{\frac{1}{2}}$ which has the t-distribution with k degrees of freedom. Here L and Q are, respectively, linear and quadratic forms in the observations and are independent of the population parameters. Sufficient conditions for such a quotient are: (1) L and Q are independently distributed; (2) $E(L)=\delta$; (3) Q/σ^2 has the χ^2 distribution with k degrees of freedom, where σ^2 is the variance of L. The author proves that (2) and (3) are incompatible with condition (4): the quotient is symmetric, that is, is unchanged by permutations of the x's among themselves and of the y's among themselves.

A. M. Mood (Princeton, N. J.).

Malmquist, K. G. The elimination of the effects of accidental errors of measurement in statistical investigations. Ark. Mat. Astr. Fys. **27A**, no. 24, 13 pp. (1941). [MF 12026]

Eddington [Monthly Not. Roy. Astr. Soc. **100**, 354–361 (1940); these Rev. **2**, 233] has treated the problem of determining the distribution function of a variate X from the distribution function of the "observed values" $X+\epsilon$, assuming that the errors of measurement ϵ are Gaussian and independent of X. The author shows that Eddington's results are equivalent to results obtained by himself [Meddelanden Astr. Obs. Lund Ser. II, no. **22** (1920); Ark. Mat. Astr. Fys. **16**, no. 23 (1922)] in treating the problem: if the absolute magnitudes of stars are distributed according to a Gaussian law and independent of the distance, what is the distribution of apparent magnitudes? Various improvements of the technique are given. *W. Feller.*

Malmquist, K. G. On some formulas for the computation of space densities. Ark. Mat. Astr. Fys. **29B**, no. 8, 7 pp. (1943). [MF 12018]

This paper contains simplifications and improvements of the methods referred to in the preceding review.

W. Feller (Ithaca, N. Y.).

Rajalakshman, D. V. On the interval between the ranked individuals of samples taken from a rectangular population. J. Madras Univ. Sect. B. **15**, 31–44 (1943). [MF 12319]

The distribution function of the interval mentioned in the title is obtained by using hypergeometric series.

W. Feller (Ithaca, N. Y.).

v. Mises, R. On the classification of observation data into distinct groups. Ann. Math. Statistics **16**, 68–73 (1945). [MF 12359]

The problem considered is that of deciding from which of a finite number of completely specified populations a sample has been drawn. The decision is made according to the principle of maximizing the minimum expected proportion of correct decisions (minimizing the maximum risk in Wald's terminology, when all errors have the same weight). The partition of the sample space that solves the problem is characterized and some examples are worked.

H. Scheffé (Princeton, N. J.).

Mann, Henry B. On a problem of estimation occurring in public opinion polls. Ann. Math. Statistics **16**, 85–90 (1945). [MF 12361]

Baten, William Dowell. Analyzing degrees of freedom into comparisons when the "classes" do not contain the same number of items. Nat. Math. Mag. **19**, 221–228 (1945). [MF 12438]

Radhakrishna Rao, C. On the linear set up leading to intra and inter block informations. Science and Culture **10**, 259–260 (1944). [MF 11972]

It is well known that the yields of different varieties in the same block in an experiment in randomized blocks are not independent of each other. The author outlines a method of analysis which takes account of this fact. He promises that a more detailed discussion of his method will appear shortly. *H. B. Mann* (Providence, R. I.).

Cornish, E. A. The recovery of inter-block information in quasi-factorial designs with incomplete data. I. Square, triple, and cubic lattices. Commonwealth of Australia. Council Sci. Ind. Res. Bull. no. **158**, 22 pp. (1943). [MF 12456]

Cornish, E. A. The recovery of inter-block information in quasi-factorial designs with incomplete data. II. Lattice squares. Commonwealth of Australia. Council Sci. Ind. Res. Bull. no. **175**, 19 pp. (1944). [MF 12458]

The author gives without derivation approximate formulae for tests of significance in square, triple and cubic lattices [first paper] and lattice squares [second paper] in the case where values are missing and it is desired to use the inter-block information. *H. B. Mann* (Providence, R. I.).

Phipps, I. F., Pugsley, A. T., Hockley, S. R., and Cornish, E. A. The analysis of cubic lattice designs in varietal trials. Commonwealth of Australia. Council Sci. Ind. Res. Bull. no. **176**, 41 pp. (1944). [MF 12457]

This paper contains a complete treatment of the analysis of cubic lattice designs. The complete analysis of an experiment on 125 varieties arranged in a $5\times5\times5$ cubic lattice design exemplifies the procedure. *H. B. Mann.*

Mathematical Economics

***von Neumann, John, and Morgenstern, Oskar. Theory of Games and Economic Behavior.** Princeton University Press, Princeton, New Jersey, 1944. xviii+625 pp. $10.00.

The book presents a new approach to the problem of rational behavior. The view taken is that the problem of rational behavior cannot be regarded as a simple maximum problem, but is of a type that arises in the theory of games of strategy. The main reasons for this attitude may be briefly summarized as follows. An essential characteristic of a social-economic organization is that the participants enter into relations with each other and the result for each participant depends not only on his own actions but also on the actions of others. For example, in a social exchange economy where several persons exchange goods, the increase of utility a participant can achieve will depend on the behavior of all the other participants. In general, the situation that arises may be characterized as follows. Each participant wishes to maximize a certain function of several variables. He controls, however, only a partial set of these variables, while the values of the remaining variables depend on the actions of the other participants, who may have different or even opposite interests. The problem of a rational choice of the variables for each participant cannot be regarded as a simple maximum problem. Instead, we have a peculiar mixture of several conflicting maximum problems. Problems of the same type arise in the theory of games of strategy.

After a discussion of the nature of the problem of economic behavior, the authors turn to a development of the theory of games. The main features of the rigorous definition of the concept of a game are as follows. A game consists of a sequence of moves, $M_1, \cdots, M_\nu$. Each move M_k consists of a number α_k of alternatives, denoted by $A_k(1), \cdots, A_k(\alpha_k)$. There are two kinds of moves, personal and chance moves. The move M_k is a personal move if any of $A_k(1), \cdots, A_k(\alpha_k)$ can be freely chosen by one of the players

specified by the rules of the game. The move M_k is a chance move if the choice of a particular alternative is made by a chance mechanism constructed in such a way that the probability of choosing any particular alternative is equal to a value specified by the rules of the game. Denote by σ_k the alternative chosen in the move M_k. The alternatives $A_k(1), \cdots, A_k(\alpha_k)$ and the number α_k may depend on the choices $\sigma_1, \cdots, \sigma_{k-1}$ in the preceding moves. Similarly, whether M_k is a personal or a chance move, the player who makes the choice if M_k is a personal move, and the probabilities assigned if M_k is a chance move, may depend on $\sigma_1, \cdots, \sigma_{k-1}$. Furthermore, the total number ν of moves may itself depend on the choices σ_i, etc. The rules of the game also specify the outcome of the play (the amount each player gets when the play is completed) as a function of $\sigma_1, \cdots, \sigma_\nu$. A game is called an n-person game if the number of players is n. Denote by $F_i(\sigma_1, \cdots, \sigma_\nu)$ the outcome of the play for player i $(i=1, \cdots, n)$. A game is called a zero-sum game if $\sum_{i=1}^{n} F_i(\sigma_1, \cdots, \sigma_\nu)=0$ identically in $\sigma_1, \cdots, \sigma_\nu$. It is shown later that the general game can be reduced to a simple normalized form which is equivalent to the original formulation of the game. This reduction is made possible by considering the strategies of the players instead of the individual moves. By a strategy of a player is meant a complete plan which specifies what choices he should make in every possible situation. Since the number ν of moves is assumed to be bounded, the number of possible strategies for player i is finite, say β_i. Then the choice of a strategy by player i is characterized by the choice of a value of a variable τ_i which can take the integral values $1, \cdots, \beta_i$. The expected value $K_i(\tau_1, \cdots, \tau_n)$ of the outcome of the game for each player can be shown to be a single-valued function of the strategies chosen by the n players. The normalized form of a general n-person game is then as follows: n functions $K_1(\tau_1, \cdots, \tau_n), \cdots, K_n(\tau_1, \cdots, \tau_n)$ and n positive integers $\beta_1, \cdots, \beta_n$ are given. The variable τ_i can take the integral values $1, \cdots, \beta_i$ $(i=1, \cdots, n)$. Player i chooses a particular value of τ_i and this choice is made in complete ignorance of the choices of the other $n-1$ players. After the values $\tau_1, \cdots, \tau_n$ have been chosen, player i receives the amount $K_i(\tau_1, \cdots, \tau_n)$. All further discussions are based on this normalized form of a game. The zero-sum restriction now means that $\sum_{i=1}^{n} K_i(\tau_1, \cdots, \tau_n)=0$ identically in $\tau_1, \cdots, \tau_n$.

The discussion of zero-sum two-person games provides a foundation for the general theory of n-person games. Denote by $K(\tau_1, \tau_2)$ the outcome of the game for player 1. Then the outcome for player 2 is given by $-K(\tau_1, \tau_2)$. The fundamental problem is that of a suitable choice of the value of τ_i by player i $(i=1, 2)$. Player 1 wishes to maximize $K(\tau_1, \tau_2)$ and player 2 wishes to minimize $K(\tau_1, \tau_2)$. For any particular choice of τ_1, player 1 can be sure that he will get at least $\min_{\tau_2} K(\tau_1, \tau_2)$, but he cannot be sure that he will get more than this amount. Thus, if player 1 chooses a value τ_1' of τ_1 for which $\min_{\tau_2} K(\tau_1, \tau_2)$ takes its maximum value, he is certain to receive an amount at least $v_1=\max_{\tau_1} \min_{\tau_2} K(\tau_1, \tau_2)$. We shall refer to such a choice τ_1' as a minimax strategy of player 1; similarly, a choice of a value τ_2' of τ_2 for which $\max_{\tau_1} K(\tau_1, \tau_2)$ takes its minimum value is a minimax strategy of player 2. If player 2 follows a minimax strategy, he can be sure to receive an amount at least $-v_2=-\min_{\tau_2} \max_{\tau_1} K(\tau_1, \tau_2)$. If player 1 does not follow a minimax strategy, his gain may be less than v_1. It is shown that $v_1 \leqq v_2$ always holds. A game for which $v_1=v_2$ is called strictly determined. This case is of decisive importance, since the common value v of v_1 and v_2 may be interpreted as the value of the game (for player 1) and the minimax strategies may be regarded as good strategies. If a game is strictly determined and if each of the players follows a minimax strategy, player 1 will get the amount v and player 2 the amount $-v$. For strictly determined games the choice of minimax strategies by the two players creates a perfectly stable situation in the sense that even if one of the players were able to discover his opponent's strategy, he still would have no reason to deviate from his own minimax strategy. No such stable situation exists for games which are not strictly determined.

Thus a satisfactory solution can be given only for strictly determined games; for other games there are difficulties even in defining the meaning of a good strategy. Therefore the problem is reformulated as follows. Instead of choosing a particular value of τ_1, player 1 considers all possible values $1, \cdots, \beta_1$ and chooses merely the probabilities $\xi_1, \cdots, \xi_{\beta_1}$ with which he is going to use them. Similarly, player 2 chooses the probabilities $\eta_1, \cdots, \eta_{\beta_2}$ with which he is going to use the values $1, \cdots, \beta_2$ of τ_2. Thus, the two choices are characterized by vectors $\xi=(\xi_1, \cdots, \xi_{\beta_1})$ and $\eta=(\eta_1, \cdots, \eta_{\beta_2})$. The expected value of the outcome $K(\tau_1, \tau_2)$ is then $K^*(\xi, \eta)=\sum_{j=1}^{\beta_2}\sum_{i=1}^{\beta_1} K(i, j)\xi_i\eta_j$. The main theorem is that the game corresponding to the outcome function $K^*(\xi, \eta)$ is always strictly determined. Player i is said to have a pure strategy if he chooses a particular value of τ_i, and a mixed strategy if he chooses merely the probability distribution of the possible values of τ_i. Thus if both players are permitted to use mixed strategies, the game is always strictly determined. A number of special cases and examples are discussed.

The authors next turn to the discussion of zero-sum n-person games with $n>2$, for which the problem has essentially new features. In fact, when $n=2$ there is no possibility of coalitions, while when $n>2$ some players may find it advantageous to form a coalition against the others. The possible formation of coalitions and the compensations that may be necessary between coalition partners play a decisive role. In the general theory the formation of coalitions and the payment of compensations are governed entirely by the characteristic function of the game, defined as follows. Let S be any subset of the set I of all players, $-S$ the complement of S. Suppose that all players in S cooperate with each other, and all players in $-S$ cooperate with each other. Then each of the groups can be regarded as one player and the game reduces to a zero-sum two-person game. Thus the game will have a definite value $v(S)$ for S. The function $v(S)$ is called the characteristic function. A game is called inessential if for any subset S we have $v(S)=\sum_{i \in S} v(i)$, where i denotes the subset consisting of the single player i. In such a game nothing can be gained by forming coalitions and, therefore, the value of the game for player i is given by $v(i)$. A distribution or imputation is a set of n numbers $\alpha_1, \cdots, \alpha_n$, where α_i is the amount allotted to player i. For any inessential game the solution consists of the single imputation $v(1), \cdots, v(n)$.

A game which is not inessential is called essential. For essential games coalitions and compensations play a decisive role. A solution of an essential game will be a set of imputations rather than a single imputation; furthermore, there will be, in general, several solutions. The definition of a solution is based on the notions of effective sets and domination. A subset S of the set I of all players is said to be effective for the imputation $\alpha=(\alpha_1, \cdots, \alpha_n)$ if $\sum_{i \in S} \alpha_i \leqq v(S)$.

An imputation α dominates another imputation β if there exists a nonempty set S such that S is effective for α and $\alpha_i > \beta_i$ for all i in S. A solution is defined as a set V of imputations satisfying the following two conditions. (1) No imputation in V is dominated by another imputation in V. (2) Any imputation not in V is dominated by at least one imputation in V. The reason that a solution V will consist of more than one imputation is that domination is not a transitive relation. A solution V may be regarded as a possible standard of rational behavior, possessing a certain inner stability which would be upset by omission of any of the two restrictions imposed on a solution. Condition (1) is needed to eliminate inner inconsistencies in V. The desirability of condition (2) can be seen as follows. For any set of imputations which satisfies (1) there will be an imputation β outside V such that β dominates an imputation α in V. The players can hardly be expected to maintain faith in V as a standard of rational behavior unless there is another imputation γ in V which dominates β. If a solution V is accepted as a standard of behavior, no imputation conforming with the accepted standard can be overruled by another. On the other hand, any imputation not conforming with the accepted standard can be overruled by an imputation conforming with this standard. There is nothing objectionable in the fact that there will usually be no unique solution. This simply means that different "established orders of society" or "accepted standards of behavior" may be built on the same physical background (the same set of rules describing a game). Each of these standards of behavior will have the characteristics of inner stability, but the different standards may well be in contradiction with each other.

The mathematical difficulties in finding solutions for games on the basis of their characteristic functions are considerable even for small n. The complexity of the problem increases rapidly with increasing n. Although there is hardly any doubt that every game has at least one solution, no general existence theorem has yet been established. A complete and exhaustive discussion is given for the essential 3-person game, for which all solutions are known. There is also a detailed discussion of zero-sum 4-person games. No systematic and exhaustive theory is available for $n>4$. However, a number of special cases are treated, in particular, the so-called decomposable games and simple games. It is shown that all solutions of decomposable games can be determined if all solutions of the constituents are known. Furthermore, a simple game is discussed for which all solutions are determined for any value of n.

In dealing with problems of behavior in a social economy, the zero-sum restriction is not very realistic. An extension of the theory to general non-zero-sum games is achieved by re-interpreting an n-person general game Γ as an $(n+1)$-person zero-sum game in the following way. Let $K_i(\tau_1, \cdots, \tau_n)$ $(i=1, \cdots, n)$ be the outcome functions of the n-person general game Γ. A fictitious player $n+1$ is introduced by defining $K_{n+1}(\tau_1, \cdots, \tau_n) = -\sum_{i=1}^{n} K_i(\tau_1, \cdots, \tau_n)$. In this way we obtain an $(n+1)$-person zero-sum game $\bar{\Gamma}$, which is called the zero-sum extension of Γ. It is shown that the only solutions of $\bar{\Gamma}$ which can be interpreted as solutions of Γ are those which are in accordance with the fact that the fictitious player cannot have even an indirect influence on the course of the game. It turns out that a solution $\bar{V}$ of $\bar{\Gamma}$ can be regarded as a solution of Γ if for any imputation $\alpha=(\alpha_1, \cdots, \alpha_{n+1})$ in $\bar{V}$ we have $\alpha_{n+1}=v(n+1)$. The solutions of all general games with $n \leqq 3$ are derived and the results are applied to a social exchange economy with 2 and 3 participants. These economic applications deserve special attention, since even in these simple cases the new approach deviates from the ordinary viewpoint. Consider first the case of two participants: 1, the seller; 2, the buyer. Suppose that the transaction is the sale of one unit A. Let u denote the value of A to 1 and v $(v>u)$ to 2. Then the commonsense result is that the price p may be anywhere between the two limits u and v. The theory of games leads to exactly the same result. However, if the conditions of the problem are changed by allowing transactions involving any or all of s units of a commodity $(s>1)$, the result implied by the theory of games will differ from the commonly accepted viewpoint as expressed, for example, in the Boehm-Bawerk theory. While the number of units transferred according to the theory of games will be the same in both theories, the Boehm-Bawerk theory restricts the price to a narrower interval. The reason is that the ordinary viewpoint assumes the existence of a unique price valid for all transfers, while the theory of games allows for premiums and rebates (compensations) which obliterate the unique price. Another qualitative discrepancy occurs in the case where one seller, 1, sells a unit A to either buyer 2 or buyer 3. Let u be the value of A for 1, v for 2 and w for 3, where $u<v \leqq w$. The solution V given by the theory of games is the sum of the following two sets of imputations: the set of all imputations $(\alpha_1, \alpha_2, \alpha_3)$ which satisfy the condition (*) $v \leqq \alpha_1 \leqq w$; $\alpha_2=0$; $\alpha_3=w-\alpha_1$, and the set of imputations for which (**) $u \leqq \alpha_1 \leqq v$; α_2, α_3 are certain monotonic decreasing functions of α_1. In the ordinary treatment only imputations satisfying (*) are considered. The imputations satisfying (**) can be interpreted as arising from a coalition of the two buyers against the seller. Also, the case of divisible goods is discussed for the 3-person market. Some remarks are made about the general n-person market.

The book will be of interest to sociologists and economists who are interested in a rigorous foundation of the theory of rational behavior, in particular, the theory of exchange in a general market. [It may be of interest to mention that the theory of games has applications to statistics as well, since the general problem of statistical inference may be treated as a zero-sum two-person game.] The authors make an endeavor to present the material in a form which will make it understandable to readers without a special knowledge of any part of advanced mathematics. In spite of this, the book is not really elementary, because of the intricate nature of many of the mathematical deductions. Nonmathematical readers will find the excellent verbal expositions given parallel with every major mathematical deduction very helpful.

A. Wald (New York, N. Y.).

Wald, Abraham. Generalization of a theorem by v. Neumann concerning zero sum two person games. Ann. of Math. (2) 46, 281–286 (1945). [MF 12402]

Let $\xi=(\xi_1, \xi_2, \cdots)$ and $\eta=(\eta_1, \eta_2, \cdots)$ denote variable vectors, with m and n components, respectively, subject to the conditions $\xi_i \geqq 0$, $\eta_j \geqq 0$, $\sum \xi_i = 1$, $\sum \eta_j = 1$; let (k_{ij}) be a fixed $m \times n$ matrix; denote the bilinear form $\sum\sum k_{ij}\xi_i\eta_j$ by $K(\xi, \eta)$; and consider the validity of the relation

$$(*) \qquad \sup_{\xi} \inf_{\eta} K(\xi, \eta) = \inf_{\eta} \sup_{\xi} K(\xi, \eta).$$

Its truth for finite m and n is of central importance in von Neumann's theory of games [cf. the preceding review], for it means that every zero-sum two-person game is "generally strictly determined." In this paper it is pointed out

that (*) may be false for m and n both infinite (let $k_{ij}=1$ for $i>j$, 0 for $i=j$, -1 for $i<j$). Under the assumption that the matrix elements k_{ij} are bounded, (*) is proved true when one of m, n is finite, and a necessary and sufficient condition for (*) is found when both m and n are infinite.

H. Scheffé (Princeton, N. J.).

Wold, Herman. A synthesis of pure demand analysis. II. Skand. Aktuarietidskr. **1943**, 220–263 (1944). [MF 11048]

[The first part appeared in the same journal **1943**, 85–118 (1943); these Rev. **5**, 130.] The author formulates five axioms concerning indifference maps. The first conclusion is that there exists a nonnegative, nondecreasing function $U(q_1, \cdots, q_n)$ defined for all points $(q_i \geqq 0)$ of the quantity space such that for each positive c the set of points satisfying the equation $U=c$ is a locus of indifference. In the interior of the "effective" region of the quantity space any locus of indifference forms a surface which can be represented, for every index i, by an equation of the type

$$q_i = S_i(c, q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_n),$$

where c is a parameter and S_i is a single-valued continuous and decreasing function of each variable. The author introduces a novel classification of the budgets (points of the quantity space). A budget q is said to be a point of the "region of compromise" if there exists at least one balance plane $\beta(q)$ through q such that, for any n budgets $q^{(i)}$ in $\beta(q)$ which have q for compromise ($q \neq q^{(i)}$ and q is a point of the simplex determined by the $q^{(i)}$), q is preferred to all the extremal alternatives $q^{(i)}$. A budget q is a point of the region of "linear equivalence" if there exists a balance plane $\beta(q)$ through q such that, for any n budgets in $\beta(q)$ which have q for compromise, q is preferred or equivalent to all the extremal budgets, and, for at least one set of extremal budgets $\{q^{(i)}\}$, q is equivalent to each $q^{(i)}$. A point q is said to belong to the "region of either-or" if there exists at least one balance plane $\beta(q)$ through q such that $\beta(q)$ contains at least one set of budgets which have q for compromise and all of which are preferred to q.

Among other results it is shown that a balance plane contains an optimal budget if all budgets in the plane belong to the region of compromise. The results mentioned so far have been derived merely on the basis of axioms I-V with no assumptions concerning the existence of derivatives. It is shown that, if U is a strictly increasing function having continuous derivatives of the first two orders, and if each indifference surface is convex to the origin, then each balance plane contains an optimal budget. The remaining sections deal with demand functions. Using merely axioms I-V it is shown that the demand functions are continuous functions of income and prices in the "optimum" region of the space of prices and income. Under suitable differentiability conditions explicit formulas for the derivatives of the demand functions, essentially due to V. Pareto and E. Slutsky, are derived. [Cf. the following review.]

A. Wald (New York, N. Y.).

Wold, Herman. A synthesis of pure demand analysis. III. Skand. Aktuarietidskr. **1944**, 69–120 (1944). [MF 11047]

[Cf. the preceding review.] This paper discusses price and demand functions. It is shown that in a certain domain of the quantity space the price functions exist and are single-valued. Furthermore, it is shown that in the same domain they must satisfy inequalities and integrability conditions which were stated by Hotelling [Econometrica **3**, 66–78 (1935)]. [The author refers to an earlier paper of Hotelling where only the integrability conditions are mentioned.] An interesting result is stated in a theorem the essence of which may be expressed as follows. The assumption that the individual demand functions have continuous derivatives of the first two orders and that their Jacobian is different from zero everywhere is self-contradictory if the balance relation, the proportionality relation and the conditions mentioned above are not fulfilled everywhere. On the other hand, if the above relations are fulfilled everywhere, there is an indifference map which satisfies axioms I-V and which reproduces the given demand functions. The implication is that the demand-function approach which postulates the existence of demand functions is no more general than the approach based on indifference maps where the demand functions are derived from the underlying indifference map.

The remainder of the paper deals with demand elasticities. In an appendix some remarks are made on the statistical methods for estimating elasticities. The author finds that the theoretical relations among elasticities are more nearly fulfilled for statistical estimates if they are obtained by taking the regression of $Y=\log$ expenditure (or log demand) on $X=\log$ income (or log price) than by taking the regression of X on Y, or some intermediate alternative, such as the orthogonal or diagonal regression. *A. Wald.*

Marschak, Jacob, and Andrews, William H., Jr. Random simultaneous equations and the theory of production. Econometrica **12**, 143–205 (1944). [MF 11417]

The idea is stressed that for purposes of estimation and prediction it is frequently necessary to set up a simultaneous system of equations describing the interrelations of all the relevant economic variables, because the economist usually cannot choose a "dependent" variable while keeping the other (independent) variables under control. The estimation of the unknown parameters in the system is then to be made by using statistical methods appropriate to simultaneous equations. This new approach is applied to P. H. Douglas's measurements of production functions. The theory of production under imperfect competition is reformulated by considering the differences between firms with respect to their technological or economic efficiency as "random fluctuations." *A. Wald* (New York, N. Y.).

***Mosak, Jacob L. General-Equilibrium Theory in International Trade.** The Cowles Commission for Research in Economics, Monograph no. 7. Principia Press, Inc., Bloomington, Ind., 1944. xiii+187 pp. $2.50.

This is an attempt to generalize the classical theory of international trade and harmonize it with the general-equilibrium theory of value for a closed economy. The work is divided into two parts. Part I is essentially a mathematical restatement of the now well-known static equilibrium analysis as developed by Hicks and others and an extension of this analysis to international trade. It will thus suffice, by way of example, to consider the equilibrium of exchange and the equilibrium of production in an international economy. Pure competition is assumed throughout. For the equilibrium of exchange the absence of unilateral payments, impediments to trade and purely domestic commodities is provisionally assumed. Let there be $n+1$ goods exchanged among n countries. Since one equation is constrained by the budget condition, the n equations sufficient

to determine the n independent equilibrium price ratios y_s (prices in terms of a standard commodity) are given by $\sum_{t=1}^{n} I_s^{(t)}=0$, $s=1, 2, \cdots, n$, where $I_s^{(t)}=X_s^{(t)}-\bar{X}_s^{(t)}$ is the import demand (or export supply if negative) for commodity s and country t and the X_s and the $\bar{X}_s$ are the consumer demand functions and the initial supplies of s. The (perfect) stability conditions are

$$\partial\left(\sum_{t=1}^{n} I_r^{(t)}\right)\Big/\partial y_r<0, \quad \partial\left(\sum_{t=1}^{n} I_r^{(t)}\right)\Big/\partial y_s=0, \qquad s\neq r;\ s=1, 2, \cdots, n.$$

Define $a_{sr}\equiv\partial(\sum_{t=1}^{n} I_s^{(t)})/\partial y_r$, put $J=|a_{sr}|$ and let J_{sr} be the cofactor of a_{sr} in the determinant J. Then

$$\partial\left(\sum_{t=1}^{n} I_r^{(t)}\right)\Big/\partial y_r=J/J_{rr}<0,$$

and the stability conditions are that the principal minors of successive orders of $J=|a_{sr}|$ must be alternately negative and positive. The statement of the equilibrium conditions is modified as the initial restrictions are removed. The conditions for the theory of production are analogous. For equilibrium of the international economy the n independent equilibrium equations are of the form $\sum_{t=1}^{n} I_s^{(t)}=0$, $s=1, 2, \cdots, n$, where $I_s=X_s-\bar{X}_s-\bar{\bar{X}}_s$, with X_s the consumer demand functions for commodities in each country, $\bar{X}_s$ the initial supplies in each country, and $\bar{\bar{X}}_s$ the entrepreneurial supply functions for products and demand functions for factors. These functions are determined in the normal way. The X_s and $\bar{\bar{X}}_s$ are homogeneous of degree zero.

Part II is a restatement of the Hicksian theory of dynamic equilibrium and an extension of this analysis to international trade. For a closed economy the first-order conditions for intertemporal market equilibrium and the stability conditions in the current period run parallel to those for market equilibrium in a static economy except that securities and money must be included. The equilibrium conditions are that the excess market demands for each of the n commodities (product or factor), m securities, and money must be zero. All excess demands are functions of current prices and interest rates. Since one equation is under constraint, $m+n$ equations are sufficient to determine the n current prices of commodities and the m current interest rates. The conditions for dynamic market equilibrium in an international economy are analogous to those for a closed economy. The effects of varying different parameters of the equilibrium system are discussed for all cases. *M. P. Stoltz* (Providence, R. I.).

***Lange, Oscar. Price Flexibility and Employment.** The Cowles Commission for Research in Economics, Monograph no. 8. Principia Press, Inc., Bloomington, Ind., 1944. ix+114 pp. $2.00.

The author is concerned with effects of price flexibility upon employment and economic stability. The key to this problem is found to lie in the substitution between goods and money. The price of a good is defined as flexible if it falls whenever it is in excess supply and rises whenever it is in excess demand. In a flexible price economy let there be excess supply of a factor of production. Its price will fall. If the prices of other factors and the prices of products fall less than in proportion to the price of the factor or to marginal cost, there will be an increase in the quantity demanded of the factor via a substitution effect and an expansion effect. The condition for this price behavior is that a proportional fall in all prices (interest rates constant) will reduce the excess demand for cash balances to the extent that substitution of goods for money takes place. The reaction of the community to a proportional price change is called the monetary effect. Since substitution of goods for money takes place when the excess demand for cash balances changes more than in proportion to the change in prices, the monetary effect is positive when real excess demand for cash balances diminishes as prices fall and increases as prices rise. The real excess demand depends on two factors: (1) the real demand for cash balances, and (2) the real quantity of money. The real demand for cash balances varies in the same direction as current prices, in the opposite direction, or not at all according as the price expectations are prevailingly inelastic, prevailingly elastic, or of unit elasticity. If the ratio of the increment in the real quantity of money (ΔM) to the increment in the real demand for cash balances (ΔD) is defined as the "responsiveness of the monetary system" (if $\Delta D=0$, responsiveness is measured by ΔM), then the conditions for a positive monetary effect are that elastic price expectations require $\Delta M/\Delta D>1$, inelastic price expectations require $\Delta M/\Delta D<1$, and static expectations require $\Delta M/\Delta D>1$ when prices are falling and $\Delta M/\Delta D<1$ when prices are rising.

The effect of a change in the current price of a commodity upon its current demand and supply depends jointly on intertemporal substitution and on intratemporal substitution and expansion. Intertemporal substitution depends on the elasticity of expectations of prices of the commodity and its substitutes and complements, while intratemporal substitution and expansion depend on the monetary effect and the elasticity of expectation of price of other goods. With a given monetary effect, elastic price expectations exert a destabilizing influence, while inelastic price expectations exert a stabilizing influence. If the price expectations are subject to uncertainty, the planning period is shortened and the effect of intertemporal substitution on current purchases and sales is reduced. This implies a weakening of the monetary effect and an active monetary policy to strengthen it may be necessary to assure stability. The removal of the assumption of pure competition requires some modification of these results. The modifications are definitional in the case of simple monopoly or monopsony, but oligopoly and oligopsony may retard the substitution and expansion effects because of irresponsiveness of output or input or because a positive monetary effect may be absorbed in an increase in the degree of control. Similarly, the influence of international trade on a flexible price economy may be stabilizing or destabilizing, the result depending on the net impact on the real quantity of money.

The method of analysis is applied to the problems of changes in the propensity to consume and in the returns on investment, and to innovations, with the conclusion that the classical doctrine that price flexibility results in automatic maintenance of equilibrium is valid only under very special conditions. The author sets out recommendations for planning policy. A useful set of mathematical notes on the stability of equilibrium is appended. *M. P. Stoltz.*

Mathematical Biology

Wilson, Edwin B., and Worcester, Jane. The epidemic curve with no accession of susceptibles. Proc. Nat. Acad. Sci. U. S. A. 30, 264–269 (1944). [MF 11039]

The equation for the number x of susceptibles, referred to an initial number $x=1$ at $T=0$, is $\log[x(T+1)]-x(T)=k$,

where T is proportional to the time and the constant k is determined in such a manner that the maximum rate of infectious cases, that is, the maximum of $(-dx/dT)$, is equal to a given number z. Expanding x in power series in T the authors obtain k as a power series in z. The calculations are carried out up to terms of fifth order in $x(T)$, which correspond to terms of third order in $k(z)$. The result is compared with previous expressions of the epidemic curve [same Proc. **28**, 361–366 (1942); **29**, 43–48 (1943); **30**, 37–44 (1944); these Rev. **4**, 201; **6**, 12].

I. Opatowski (Chicago, Ill.).

Geppert, Maria-Pia. Mathematische Theorie der Zeit/Menge-Beziehungen bestimmter Vergiftungsvorgänge. Z. Angew. Math. Mech. **23**, 269–278 (1943). [MF 11750]

Empirical curves showing certain observations connected with poisoning processes have the shape of hyperbolas. The author considers mathematical assumptions which would "explain" this fact. The mathematical content is elementary.

W. Feller (Providence, R. I.).

García, Godofredo. On the integration of the complete equation of diffusion and its application to the study of cells. Actas Acad. Ci. Lima **7**, 255–287 (1944). (Spanish) [MF 11856]

Identical with a paper in Revista Ci., Lima **46**, 293–325 (1944); these Rev. **6**, 12.

García, Godofredo. On the integration of the equation of diffusion in the case of a spherical cell. Revista Ci., Lima **46**, 417–450 (1944) = Actas Acad. Ci. Lima **7**, 317–350 (1944). (Spanish) [MF 11391]

The author obtains solutions in terms of Legendre polynomials for the problem of the diffusion of a metabolite, autocatalytically produced in a spherical cell. Diffusion is assumed to be slow. The paper is a continuation of a previous one by the same author [same Revista **46**, 293–325 (1944); these Rev. **6**, 12]. Both papers are quite obscure, but the boundary condition seems to require a vanishing concentration of the metabolite at the cell's exterior.

A. S. Householder (Washington, D. C.).

Rosenblatt, Alfred. Integration, for the case of a spherical cell, of the partial differential equation of diffusion encountered by the regular member of the Academy, Mr. Godofredo García. Revista Ci., Lima **46**, 401–415 (1944) = Actas Acad. Ci. Lima **7**, 301–315 (1944). (Spanish) [MF 11501]

[Cf. the preceding review.] The equation in question is

$$\partial\Gamma/\partial t = D\Delta\Gamma + \{A + BE(t)\}\Gamma,$$

Γ representing the concentration of a metabolite autocatalytically produced. With García, the author sets

$$\Gamma = e^{\int [A+BE]\,dt} H(x, y, z, t)$$

and proceeds to the decomposition

$$H = \sum S_i(r)\Theta_i(\theta)\Phi_i(\phi)R_i(t).$$

As a result the R_i are exponentials, Φ_i expressible in sines and cosines, the Θ_i in Legendre functions, and the S_i in Bessel functions. In illustration, consideration is given to the case where Γ vanishes at the surface of the cell and is initially equal to a prescribed point function.

A. S. Householder (Washington, D. C.).

RELATIVITY

Weyl, Hermann. How far can one get with a linear field theory of gravitation in flat space-time? Amer. J. Math. **66**, 591–604 (1944). [MF 11397]

The author constructs the differential operators which could be used in a linear field theory of gravitation within the framework of special relativity. If it is assumed that the field equations imply the law of conservation of energy and momentum in the simple form $\partial T_i^k/\partial x^k = 0$, the possible operators reduce to two cases, a regular one, which is Einstein's theory of weak fields, and a singular one. The regular case resembles closely Maxwell's theory of the electromagnetic field. It is shown how the mechanical laws may be derived by describing a particle in terms of the surrounding "local" field. G. D. Birkhoff's theory [Proc. Nat. Acad. Sci. U. S. A. **30**, 324–334 (1944); these Rev. **6**, 72; cf. also the following review] is discussed and it is shown that he avoids using the special operators which preserve the conservation laws. In this way the bond between the mechanical and the field equations, which is a feature of general relativity, is destroyed.

G. C. McVittie (London).

Birkhoff, George D. The mathematical concept of time and gravitation. Bol. Soc. Mat. Mexicana **1**, no. 4 and **5**, 1–23 (1944). (Spanish) [MF 12235]

This is the lecture containing the first detailed presentation of the author's new theory of relativity. It was originally announced as forthcoming in the Proceedings of the Astrophysical Congress in Puebla, Mexico on February 20, 1942; cf. these Rev. **4**, 116, 285. Later communications on the subject have appeared in Proc. Nat. Acad. Sci. U. S. A. **29**, 231–239 (1943); **30**, 324–334 (1944); Phys. Rev. (2) **66**, 138–143 (1944); these Rev. **4**, 285; **6**, 72.

Barajas, Alberto. Birkhoff's theory of gravitation and Einstein's theory for weak fields. Bol. Soc. Mat. Mexicana **1**, no. 4 and **5**, 41–46 (1944). (Spanish) [MF 12237]

Translation of a paper published in Proc. Nat. Acad. Sci. U. S. A. **30**, 54–57 (1944); these Rev. **5**, 218.

Graef Fernández, Carlos. The motion of two bodies in Birkhoff's theory of gravitation. Bol. Soc. Mat. Mexicana **1**, no. 4 and **5**, 25–39 (1944). (Spanish) [MF 12236]

The motion of one particle under the gravitational attraction of another is worked out on the basis of Birkhoff's theory of relativity, which employs Minkowski space-time. Since gravitational effects are assumed to be propagated with the velocity of light, "retarded potentials" must be used. It is assumed that the gravitational potentials at a point distant r from a particle moving with velocity v are derivable from a function of the form $(1-v^2)^{\frac{1}{2}}/(r-R\cdot V)$, where $v=|V|$, $r=|R|$, and $R\cdot V$ is the scalar product of R and V. This function is chosen on the ground that it reduces asymptotically to $1/r$ near $r=0$. The advance per radian of the perihelion for the orbit of one particle moving about another of comparable mass is worked out. In the planetary case, the advance reduces to $3m/a(1-e^2)$ per radian, where m is the mass of the central body, a is the mean distance of the planet and e the eccentricity of its orbit. [General relativity gives the same result.] It is sug-

gested that the results have an application to the orbits of double stars. *G. C. McVittie* (London).

Vallarta, Manuel Sandoval. Physical aspects of Birkhoff's theory of gravitation. Bol. Soc. Mat. Mexicana 1, no. 4 and 5, 47–53 (1944). (Spanish) [MF 12238]
An expository lecture.

Wyman, Max. Static isotropic solutions of Einstein's field equations. Phys. Rev. (2) 66, 267–274 (1944). [MF 11418]
Metrics of the general form

$$ds^2 = V^2dt^2 - (dx^2+dy^2+dz^2)/W^2$$

are considered, where V, W are functions of x, y, z only. The permissible forms for V, W are worked out under the conditions that the components of the energy-tensor shall satisfy $T_1^1 = T_2^2 = T_3^3$ and $T_j^i = 0$ for $i \neq j$. Three cases are possible, namely, (1) $W = W(x)$, $V = V(x)$; (2) $W = xf(u)$, $V = V(u)$, $u = y/x$; (3) $W = W(r)$, $V = V(r)$, $r^2 = x^2+y^2+z^2$, where V, W, f are undetermined functions. Other expressions for these functions can, however, be obtained by using coordinate-transformations. When all the components of the energy-tensor are zero, V, W can be found as explicit functions of the coordinates and are given for the cases when the cosmical constant λ is zero and when it is not. One of the metrics for the case $\lambda = 0$ is the Schwarzschild metric; for $\lambda \neq 0$, the functions V, W are elliptic functions. *G. C. McVittie* (London).

Cheng, Kai-Chia. A simple calculation of the perihelion of Mercury from the principle of equivalence. Nature 155, 574 (1945). [MF 12497]
The relativistic equation of a planetary orbit is obtained by applying the "principle of equivalence" to the classical equations of motion. The author claims that the principle of equivalence implies that dr, dt must be replaced by

$$(1) \qquad (1+\lambda/r)dr, \quad (1+\lambda/r)^{-1}dt,$$

where $\lambda = GM/c^2$. The classical Lagrangian now assumes the form $\frac{1}{2}(\beta^{-1}\dot{r}^2+\beta^2r^2\dot{\theta}^2)+GM/r$, where $\beta = 1+\lambda/r$. Writing the usual Lagrangian equations and retaining only terms of the first order in λ, the well-known relativistic equation of the orbit is finally obtained:

$$(2) \qquad d^2\mu/d\theta^2+\mu = GM/h^2+3\lambda\mu^2, \qquad \mu = 1/r.$$

It is not clear to the reviewer how (1) can be arrived at without a previous knowledge of Schwarzschild's line-element. If, however, the line-element is known, then the author's calculations are similar to, and no simpler than, those required to derive (2) rigorously from the equation of a geodesic. *A. Schild* (Toronto, Ont.).

Schild, A. On Milne's theory of gravitation. Phys. Rev. (2) 66, 340–342 (1944). [MF 11533]
Milne's equations for the motion of two gravitating bodies are written in tensor form. These equations can be used to determine the motion only if an additional assumption is made by which each point on the world-line of one body is associated with a point on that of the other. The association is performed by Milne by using his definition of simultaneity. However, since this definition is not invariant under Lorentz transformations, neither is his gravitational theory. The latter can be made so by, for example, making the association through the retarded potential but this introduces a new ad hoc hypothesis. The foregoing criticism fails if the world-line of one of the bodies is a straight line through the space-time origin and it is only to this case that Milne's gravitational theory applies in a Lorentz invariant form. *G. C. McVittie* (London).

Moghe, D. N. Kinematical theory and general relativity. J. Univ. Bombay (N.S.) 11, part 3, 1–5 (1942). [MF 12182]
Some properties of the curvature tensor of the space with metric

$$ds^2 = F^2\{dt^2 - c^{-2}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)\}$$

are examined in the cases where F is a function of $X = t^2-r^2$ only and where F is a function of X and of

$$\xi = (t-c^{-2}\textstyle\sum x\dot{x}/dt)^2/X(1-c^{-2}\sum(dx/dt)^2).$$

In the second case, the author makes the mistaken assumption that the space is Riemannian. *A. Schild.*

Moghe, D. N. A note on isotropic manifolds. J. Univ. Bombay (N.S.) 11, part 3, 6–8 (1942). [MF 12183]
For the line-element $ds^2 = dt^2+g_{ij}dx^idx^j$ the condition of isotropy implies that the principal components of the energy tensor are equal. The equations of geodesics and an expression for the geodesic curvature are obtained. *A. Schild.*

Mariani, Jean. Espaces de Riemann à torsion et électromagnétisme. C. R. Acad. Sci. Paris 218, 447–449 (1944). [MF 12115]
A generalization of Einstein's gravitational field theory is suggested, starting from an asymmetric affine connection in a Riemannian space. The skew-symmetric part of the connection is used to interpret the electromagnetic field and a generalized Einstein tensor is introduced for the equation of the gravitational field. The equations for a perfect fluid of charged matter are derived. *S. Chern.*

Hély, Jean. Loi synthétique de la gravitation et de l'électromagnétisme. Ann. Physique 19, 208–214 (1944). [MF 12297]
The author develops a gravitational theory on the Minkowskian space-time background of restricted relativity. The theory, which is invariant under Lorentz transformations, is easily extended to include electromagnetic phenomena. The postulated law of motion yields many conclusions which are usually deduced from Einstein's general theory of relativity. In particular, the author obtains expressions for the advance of perihelion of planetary orbits, the deflection of light rays by massive bodies, the red-shift of atomic spectra in strong gravitational fields and the proportionality of inertial and gravitational mass.
The law of motion of a particle of mass m and charge e is expressed in the form of a variational principle: the integral $\int dS$ along the worldline of the particle is stationary (more specifically, a maximum), where

$$dS = \{\cos(\phi_i+\phi_s)\}^{\frac{1}{2}}ds + e(mc^2)^{-1}(\cos\pi/6)^{\frac{1}{2}}d\mathbf{s}\cdot\boldsymbol{\psi}.$$

The "external gravitational phase" ϕ_s is a dimensionless, retarded-potential type function determined by the distribution and velocity of matter in the universe; the "internal gravitational phase" ϕ_i is a constant equal to $\pi/6$ for a material particle and to $\pi/2$ for a photon (for which e/mc^2 is taken to be zero); $\boldsymbol{\psi}$ is the electromagnetic 4-vector potential and is similar in construction to the scalar ϕ_s.
A. Schild (Toronto, Ont.).

Jonsson, Carl Victor. Dirac's wave equation in five-dimensional relativity theory. Ark. Mat. Astr. Fys. **28B**, no. 17, 7 pp. (1942). [MF 12325]

The author considers a wave equation for the electron in homogeneous five-dimensional relativity theory. The wave equation, which is invariant under coordinate- and spin-transformations, is similar to that suggested by Pauli [Ann. Physik (5) **18**, 305–336 (1933)], but differs from it by a term which is too small for experimental verification. In Pauli's equation the mass m of the electron appears as the difference of two terms, each of order $10^{29}\,m$; this is avoided in the author's formulation. In the absence of gravitation, the wave equation differs from the ordinary Dirac equation by a small term, proportional to the square root of Einstein's gravitational constant. Transformations from the homogeneous to Kaluza's older, nonhomogeneous, five-dimensional relativity theory are given, and the wave equation is obtained in the nonhomogeneous form. *A. Schild.*

Infeld, L., and Schild, A. A note on the Kepler problem in a space of constant negative curvature. Phys. Rev. (2) **67**, 121–122 (1945). [MF 11967]

Schrödinger's equation is formally generalized to a space of constant negative curvature. It is then proved that if the radius of the space is 10^{28} cm. the number of discrete energy levels is of the order of 10^{18}. The highest energy level may be positive or negative but all the rest are negative. A continuous spectrum can also exist. The calculations were repeated for the Gordon-Klein equation but the general character of the solution was not changed.

G. C. McVittie (London).

Reichenbächer, Ernst. Der Doppler-Effekt im allgemeinen Feld. Z. Astrophys. **22**, 230–235 (1943). [MF 11857]

If a particle moves in a gravitational field, then the change in its wave length depends on the radial velocity v' and on g_{44}. These two well-known influences can be expressed in one formula

$$\left(\frac{\lambda}{\lambda'}\right)^2=\frac{g_{44}(1+v'/c')}{g'_{44}(1-v'/c')},$$

where the primes refer to a point from which signals are sent. The distinction between these two influences depends, of course, on the coordinate system, as the author shows by a simple example. *L. Infeld* (Toronto, Ont.).

Bogorodsky, A. Th. Doppler effect in a static gravitation field. Acad. Sci. USSR. Poulkovo Observatory Circular no. **28**, 52–58 (1939). (English. Russian summary) [MF 11853]

It is proved in detail that, in a static gravitational field with metric $ds^2=g_{ik}dx_i^2$, the Doppler effect for light of wavelength λ is given by

$$\frac{\lambda+\delta\lambda}{\lambda}=\frac{1-(v_1/V_1)\cos(v_1,\,V_1)}{1-(v_2/V_2)\cos(v_2,\,V_2)}\left\{\frac{V_2^2-v_2^2}{V_1^2-v_1^2}\right\}^{\frac{1}{2}},$$

where $v_\sigma\cos(v_\sigma,\,V_\sigma)$ $(\sigma=1,\,2)$ are the projections of the velocities of the source and the observer on the directions of the velocities of light V_σ $(\sigma=1,\,2)$ at the source and the observer, respectively. *G. C. McVittie* (London).

Bogorodsky, A. Th. On the nature of the red shift in the spectra of extragalactic nebulae. Acad. Sci. USSR. Poulkovo Observatory Circular no. **29**, 5–19 (1940). (Russian. English summary) [MF 11854]

It is assumed that a photon corresponds to a distribution of gravitating matter (unspecified in detail) and that the energy of the photon is calculable by a second-order solution of Einstein's gravitational equations for the corresponding field. If, moreover, the photonic mass (and therefore the energy of the photon) depends on its frequency, the theory gives a change of frequency as the photon moves through space. The precise formula can be made identical with Hubble's law of red-shift by the additional assumption that the frequency-change equation shall be invariant under Lorentz transformations. *G. C. McVittie* (London).

Gregory, Christopher. On a supplement to the field equations with an application to cosmology. Phys. Rev. (2) **67**, 179–184 (1945). [MF 12215]

In addition to the field equations of general relativity

$$-8\pi g_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}$$

the supplemental equations

$$-8\pi k g_{\mu\nu}=\bar{R}_{\mu\nu}-\tfrac{1}{2}\bar{R}g_{\mu\nu}+\Lambda_1 g_{\mu\nu}$$

and, as condition for "empty" space,

$$g_{\mu\nu}=\frac{8\pi k-\Lambda}{8\pi-\Lambda_1}g_{\mu\nu}$$

are postulated. In these equations, $R_{\mu\nu}$ and $\bar{R}_{\mu\nu}$ are the contracted curvature tensors of Riemannian spaces whose metric tensors are $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, respectively. The three equations are applied to the Robertson-Weyl cosmological model

$$ds^2=dt^2-(1+\tfrac{1}{4}r^2R_0^{-2})^{-2}e^{g(t)}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)$$

and two solutions for $g(t)$ are obtained. Expressions for the red-shift are computed and, in both cases, it is found that, for small Λ and small curvature R, the proper density of energy T_4^4 is connected with the recession factor k by the relation $8\pi T_4^4=3k^2$. *A. Schild* (Toronto, Ont.).

Dingle, Herbert. A relativistic theory of temperature radiation. Philos. Mag. (7) **35**, 499–518 (1944). [MF 11341]

The main question considered in this paper is whether or not the reasons which led to the relativity theory of motion also apply to the theory of radiation. Several similarities between the two theories are given and a start is made on a relativity theory of radiation. It is also pointed out that the theory as given applies only to radiation at constant temperature and that further generalization is desirable.

M. Wyman (Edmonton, Alta.).

MECHANICS

Noronha, F. J. On reciprocal screws. Math. Student **12**, 73–74 (1945). [MF 12485]

Artobolevsky, I. I. On evolutes of connecting-rod curves. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 99–101 (1944). [MF 12574]

Dobrovolsky, V. V. Spherical three-bar curve. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 475–477 (1944). (Russian. English summary) [MF 12586]

The author deduces the equation of the three-bar curve for a spherical four-bar linkage. It is a curve of the 8th

order. The properties of the double points of the curve are also established. *Author's summary.*

Wikander, O. R. Dynamics of an elastic bar. J. Appl. Mech. **12**, A-101–A-106 (1945). [MF 12467]

The dynamics of an elastic bar with slack is of interest because of its analogy to the dynamics of a long train. With this application in mind, the author considers various dynamical systems consisting of an elastic bar, with or without slack, either with free ends or with a heavy particle (corresponding to the locomotive) attached to one end. The development is unsatisfactory because results obtained in the analysis of special problems are applied in the study of more general problems without re-examination as to their validity. However, the reviewer has detected no example of an actually erroneous result. *D. C. Lewis.*

Dobronravov, V. V. On the invariance of the Hamilton-Jacobi equation in quasi-coordinates. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 249–252 (1944). [MF 11637]

It is shown that the Hamilton-Jacobi equation, as generalized by the author [same C. R. (N.S.) **22**, 477–480 (1939)] so as to utilize quasi-coordinates, is essentially the same as the classical Hamilton-Jacobi equation written down directly from the "basic" true coordinates. *D. C. Lewis.*

Dobronravov, V. V. On the integrability of Hamilton-Jacobi equation in quasi-coordinates. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 95–97 (1944). [MF 11611]

The Staeckel criterion for the integrability of the Hamilton-Jacobi equation is found to be applicable to the author's generalized Hamilton-Jacobi equation. *D. C. Lewis.*

Dobronravov, V. V. Poisson's theorem in non-holonomic coordinates. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 231–234 (1944). [MF 12037]

The definition of Poisson's bracket is generalized so as to be applicable to holonomic systems described in terms of quasi-coordinates. In connection with such systems, the generalized bracket possesses properties exactly similar to those possessed by the usual brackets in connection with systems set up in true coordinates. The author's statement that the generalization applies also to nonholonomic systems is obscured by lack of adequate proof.
D. C. Lewis (New York, N. Y.).

Lichnerowicz, André. Sur un principe de la dynamique lié à l'invariant intégral de Poincaré-Cartan. C. R. Acad. Sci. Paris **217**, 660–662 (1943). [MF 11676]

The author gives an analytic characterization of the equations of dynamics in terms of the existence of "integral relations of invariance." The main theorem, when properly generalized, is at least implicitly identical with a result of the reviewer [Bull. Amer. Math. Soc. **43**, 277–282 (1937), in particular, § 6]. *D. C. Lewis* (New York, N. Y.).

Bulgakov, B. V. Sur les transformations des équations des systèmes non-conservatifs. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 96–99 (1944). [MF 12029]

The author utilizes the Helmholtz form of Hamilton's principle to obtain the equations of motion of a nonconservative system. He indicates the transformed system of equations resulting from a general transformation of dependent variables. An application of the method is made to a perturbation problem where the original variables are replaced by new variables which are the arbitrary constants in the case of the nonperturbed problem. *N. Levinson.*

Schönberg, Mario. On a variational principle of dynamics. I. Anais Acad. Brasil. Ci. **16**, 293–310 (1944). (Portuguese) [MF 12212]

The "general variational principle of dynamics" discussed in this paper is: "The trajectories of a holonomic system are the extremals of the time integral of a function of Lagrangian coordinates, their time derivatives (not necessarily of the first order), and of the time." This statement, although called a principle of dynamics, is (as the author himself emphasizes) of purely kinematic nature since it does not involve the conceptions of mass, force and energy. It must be combined with Newton's laws or the d'Alembert principle if we want to derive the existence of a potential or Hamilton's principle. The greater part of the paper is devoted to the investigation of conditions expressed in terms of the Hamilton-Jacobi-Lie theory and equivalent to the "general principle." For instance, when the equations of motion are given in the form $q_i = P_i(p, q, t)$, $p_i = -Q_i(p, q, t)$, these conditions are that the system must admit the infinitesimal contact transformations generated by its integrals.
G. Y. Rainich (Ann Arbor, Mich.).

Astronomy

Stumpff, K. Über eine Eigenschaft des zweiten Brennpunkts der Keplerschen Bahnellipse und ihre Verwendung in der Ephemeridenrechnung. Astr. Nachr. **273**, 179–188 (1943). [MF 11862]

Making use of the second focus of the ellipse on which a planet moves, the author defines the antifocal anomaly of a planet in a way similar to the usual definition of the true anomaly. If v is the true, w the antifocal and E the eccentric anomaly, then

$$\tan \tfrac{1}{2}v \tan \tfrac{1}{2}w = \tan^2 \tfrac{1}{2}E,$$

$$\tan \tfrac{1}{2}v = \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\varphi) \tan \tfrac{1}{2}E, \quad \tan \tfrac{1}{2}w = \tan (\tfrac{1}{4}\pi - \tfrac{1}{2}\varphi) \tan \tfrac{1}{2}E.$$

These and formulas derived from them are applied to the calculation of the ephemeris. Numerical exercises illustrate the computational procedures. *V. G. Grove.*

Bilimovitch, A. Über die Anwendungen der Pfaffschen Methode in der Störungstheorie. Astr. Nachr. **273**, 161–178 (1943). [MF 11861]

Let X_i and x_i be the components of vectors $\mathbf{X}$ and $\mathbf{x}$, respectively. The scalar product $\Phi = \sum X_i dx_i = \mathbf{X} \cdot d\mathbf{x}$ is called a Pfaffian expression; if $a_{ij} = \partial X_j/\partial x^i - \partial X_i/\partial x^j$, the differential equations $\sum_{i=1}^{N} a_{ij} dx_j = 0$ $(i = 1, 2, \cdots, N)$ are called Pfaffian differential equations. They may also be written in the form

$$\operatorname{grad} \Phi - \operatorname{grad} (\mathbf{X} \cdot d\mathbf{x}) = 0.$$

It is shown that the equations of planetary motion may be written as Pfaffian equations. Pfaffian expressions for planetary perturbations are formed. The Pfaffian method combined with vector analysis affords a simple direct approach to these problems of dynamics. *V. G. Grove.*

Chazy, Jean. Sur la réduction du problème des trois corps à un système différentiel d'ordre 6. C. R. Acad. Sci. Paris **218**, 485–488 (1944). [MF 12117]

The canonical variables of Delaunay are used. The presentation is based upon the exposition by Poincaré [Les Méthodes Nouvelles de la Mécanique Céleste, vol. 1, Gauthier-Villars, Paris, 1892], to which the author adds some explanatory remarks. *D. Brouwer.*

Schürer, M. Beitrag zur Dynamik der Sternsysteme. Astr. Nachr. 273, 230–242 (1943). [MF 11864]

The author obtains by shorter methods the solutions of the equation of continuity obtained by Chandrasekhar [Astrophys. J. 90, 1–154 (1939); 92, 441–642 (1940); these Rev. 1, 61; 3, 216]. The principal new idea can be illustrated by the following example. With the transformation $\mathfrak{r}(t)=\varrho(\tau)/\phi(\tau)$, where $\tau=\int^t dt/\phi^3$, ϕ is an arbitrary function of the argument and $\mathfrak{r}$ is the position vector of a particle moving in a field of force described by a potential function $V(\mathfrak{r})$, the equation of motion takes the form

$$d^2\varrho/d\tau^2=\varphi^{-1}(d^2\phi/d\tau^2)\varrho-\varphi^{-2}\,\mathrm{grad}_\varrho\, V.$$

Accordingly, the equation of motion for ϱ can be put in Lagrangian form with the potential function

$$\Omega=-\tfrac{1}{2}\varphi^{-1}(d^2\phi/d\tau^2)\,|\varrho|^2+\varphi^{-1}V(\varrho/\phi).$$

Consequently, a solution in $\mathfrak{r}$ and t of the equation of continuity for $V(\mathfrak{r})$ leads to a solution in ϱ and t for $\Omega(\varrho,t)$; the latter solution rewritten in terms of the original variables leads to a new solution of the equation. By this, and similar transformations, the author has been able to derive most of Chandrasekhar's results and thus throw light on the meanings of the various solutions. *S. Chandrasekhar.*

Cesco, C. U., Chandrasekhar, S., and Sahade, J. On the radiative equilibrium of a stellar atmosphere. IV. Astrophys. J. 100, 355–359 (1944). [MF 11915]

[The first three papers, by Chandrasekhar, appeared in the same J. 99, 180–190 (1944); 100, 76–86, 117–127 (1944); these Rev. 6, 76, 190.] In part II the solution of the equation of radiative transfer for the "standard case" in the theory of the formation of stellar absorption lines was obtained as an infinite series. This solution is now obtained in its numerical form in the first three approximations and tables for computing the residual intensity are given. [Cf. the four following reviews.] *G. C. McVittie* (London).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. V. Astrophys. J. 101, 95–107 (1945). [MF 11916]

[Cf. the preceding review.] The equation of transfer for the case of a curved atmosphere is

$$\mu\partial I/\partial r+r^{-2}(1-\mu^2)\partial I/\partial\mu=-\kappa\rho I+\tfrac{1}{2}\kappa\rho\int_{-1}^{1}I\,d\mu,$$

where $\kappa\rho$ is a function only of r. This equation is solved by successive approximations, the right-hand side being replaced by a sum according to Gauss's formula for numerical quadratures. The result is that the equation is replaced by $2n$ linear equations in the nth approximation. The second approximation in the case $\kappa\rho\propto r^{-n}$ $(n>1)$ is fully worked out and is found to involve quadratures over Bessel functions of purely imaginary arguments. The solutions for the case $n=2$ are found numerically. The case $\kappa\rho=$constant is also worked out, since it has importance in diffusion problems in physics. *G. C. McVittie* (London).

Cesco, C. U., Chandrasekhar, S., and Sahade, J. On the radiative equilibrium of a stellar atmosphere. VI. Astrophys. J. 101, 320–327 (1945). [MF 12463]

In parts II [same J. 100, 76–86 (1944); these Rev. 6, 76] and IV [cf. the second preceding review] the nth approximation to the solution of the problem of line-formation in a stellar atmosphere under "standard" conditions was given. The explicit solution of the set of linear equations involved is now derived. Another problem considered is a solution of the equation of transfer for the ultraviolet radiation beyond the head of the Lyman series, consistent with Zanstra's theory of the hydrogen-line emission in planetary nebulae. Numerical details and various tables involved in this solution are given. *G. C. McVittie* (London).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. VII. Astrophys. J. 101, 328–347 (1945). [MF 12464]

[Cf. the preceding review.] The problem of the radiative equilibrium of a stellar atmosphere in local thermodynamic equilibrium is considered in order to investigate the effect of a continuous absorption coefficient κ_ν which depends on the frequency of the radiation. An iterative method for solving the equations of the problem by successive approximations is devised and the first and second approximations worked out in detail. In the first approximation, the solution is assumed to be that for a gray atmosphere in which κ_ν is equal to an appropriately chosen mean absorption coefficient $\bar{\kappa}$. The best way of choosing $\bar{\kappa}$ is to define it as a straight mean of κ_ν, weighted according to the net monochromatic flux of radiation of frequency ν in a gray atmosphere. The method does not provide any justification for averaging κ_ν according to the conventional Rosseland method. *G. C. McVittie* (London).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. VIII. Astrophys. J. 101, 348–355 (1945). [MF 12465]

[Cf. the preceding review.] The method described in part II [same J. 100, 76–86 (1944); these Rev. 6, 76] is applied to the solution of the equation of transfer for the radiative equilibrium of a stellar atmosphere exposed to a parallel beam of radiation, of flux πF per unit area normal to the beam, which enters the atmosphere at an angle β relative to the normal at the boundary. The general solution in the nth approximation is found and it is shown that the integrated Planck intensity B satisfies $B(0)/B(\infty)=\sec\beta/\sqrt{3}$ in every approximation. Another result obtained is that the "reflected radiation" is related in a simple way to the law of darkening in an atmosphere characterized by a constant net flux with no incident radiation. *G. C. McVittie.*

Sen, N. R. Note on pressure relations within fluid spheres in equilibrium. Bull. Calcutta Math. Soc. 36, 147–152 (1944). [MF 12414]

Some further integral theorems in the equilibrium of a star are proved [cf. S. Chandrasekhar, An Introduction to the Study of Stellar Structure, University of Chicago Press, 1939, chap. III; N. R. Sen, Z. Astrophys. 18, 124–131 (1939)]. A typical one, valid for any configuration in which the mean density interior to r does not increase outwards, is

$$\bar{P}(r)=(1/M(r))\int_0^r P(r)\,dM(r)\leqq P_c\{1-\tfrac{3}{8}M^{\frac{1}{3}}(r)/M^{\frac{1}{3}}\},$$

where $P(r)$ denotes the pressure at r; $M(r)$, the mass interior to r; P_c, the central pressure and M, the mass of the configuration. *S. Chandrasekhar* (Williams Bay, Wis.).

Kothari, D. S., and Auluck, F. C. A note on the minimum radius for degenerate stellar masses. Philos. Mag. (7) 35, 783–786 (1944). [MF 12330]

It is known that in completely degenerate stellar configurations the radius tends to zero as the mass tends to a critical value M_c [Chandrasekhar, An Introduction to the

Study of Stellar Structure, University of Chicago Press, 1939, chap. XI]. The authors point out the fact that as the critical mass is approached a new situation arises, namely that the masses of the electrons become very large (in virtue of their velocities) and that in consequence the density in the equation of hydrostatic equilibrium should be modified. They show, furthermore, that when this effect is included the singularity can be avoided. [It should be pointed out that the phenomenon under consideration is one which happens extremely near M_c. Thus for a mass which is $0.96M_c$ the electrons at the center moving with the maximum velocity corresponding to the Fermi threshold have a mass $10m_0$, where m_0 denotes the rest mass of the electron. This has to be compared with the mass of the proton which is about $1850m_0$; but it is true that, as $M \to M_c$, the effective mass of the electron must surpass this value and in fact tend to infinity. On the other hand, it would seem that the authors' use of the virial theorem $2T+\Omega=0$ is not free of contradictions under the conditions contemplated.] *S. Chandrasekhar* (Williams Bay, Wis.).

Severny, A. B. On the stability of rotating gaseous spheres. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 53–55 (1945). [MF 12548]

In this paper the vibrational stability of slowly rotating gaseous spheres is considered. The point of departure from earlier discussions of the problem is that solutions of the variational equations are sought which are expressible as expansions in terms of the parameter $\omega^2/2\pi G\bar{\rho}_0$ (where ω denotes the uniform angular velocity of rotation, $\bar{\rho}_0$ the mean density and G the constant of gravitation); only the first order terms in such expansions are retained. The case of a slowly rotating homogeneous sphere is considered in detail and indications obtained that "slow rotation makes the gaseous spheres more unstable than without rotation." *S. Chandrasekhar* (Williams Bay, Wis.).

Roy, Sunil Kumar. On the anharmonic pulsations in the pulsation theory of the Cepheid variable. Bull. Calcutta Math. Soc. **36**, 109–121 (1944). [MF 12408]

S. Rosseland's theory of the anharmonic pulsations of a gaseous star [Monthly Not. Roy. Astr. Soc. **103**, 233–243 (1943)] is discussed, retaining terms of order higher than those kept by Rosseland. It is shown that when these terms are included several interesting features deduced by Rosseland (for example, the skewness of the velocity-time curve) appear in a much less pronounced fashion. [Similar conclusions have been reached by P. L. Bhatnagar and D. S. Kothari, Monthly Not. Roy. Astr. Soc. **104**, 292–296 (1944).] *S. Chandrasekhar* (Williams Bay, Wis.).

Schoenberg, E. Zur Dynamik der Jupiteratmosphäre. Astr. Nachr. **273**, 113–123 (1942). [MF 11860]

The paper is an attempt to explain the observed velocity gradient of the cloud formations on Jupiter's surface. Helmholtz's theory of vortices is used to investigate "stable" motion along the parallels of latitude. The condition of stability at surfaces of discontinuity is expressed by a pair of inequalities relating the latitude, the pressure gradient, the "potential" temperatures and the rotational moment of the fluid element. Adiabatic lapse rate is assumed and the oblateness of the planet is taken into account. The latter enters through the ratio $(C-A)/Ma^2$, which is known from the perturbations of the line of apsides and the line of nodes in the orbits of Jupiter's satellites. The present uncertainty in the temperature distribution precludes a complete verification of the theory. *B. Garfinkel* (New Haven, Conn.).

Ambarzumian, V. A. On the theory of brightness fluctuations in the Milky Way. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 223–226 (1944). [MF 12035]

The author derives a distribution function $P(I)$ for the probability that the total brightness in a given direction in the plane of the galactic equator will exceed I. He starts from the following assumptions. (1) Every elementary volume dV radiates in a unit solid angle an amount of energy ηdV. (2) Absorbing clouds fill space so that the number of clouds intersected by an arbitrary segment of length x is subject to the Poisson distribution with parameter νx. (3) If the light of a star passes through a cloud, the constant portion $1-q$ of the light is absorbed. It follows easily that $P(I)$ satisfies the equation

$$\nu P(I)+\eta P'(I)=\nu P(I/q),$$

from which the moments of $P(I)$ can be computed. It is also shown that assumption (3) can easily be relaxed.

W. Feller (Ithaca, N. Y.).

Hydrodynamics, Aerodynamics, Acoustics

Oseen, C. W. Contributions à la théorie analytique des marées. II. Ark. Mat. Astr. Fys. **28A**, no. 10, 43 pp. (1942). [MF 12328]

In an earlier work [same Ark. **25A**, no. 24 (1937)] the author gave the fundamental tensor F of the system of equations of motion of a viscous incompressible fluid referred to a set of axes turning with angular velocity Ω about an axis. An expression for this function F can be found by expanding the integral

$$\int_0^\pi \sin\theta \cos^{2n-2p+2p_1}\theta e^{-X^2\cos^2\theta}d\theta,$$

where $X=\frac{1}{2}R\{\mu'(t-\tau)\}^{-\frac{1}{2}}$. The substitution $x=X\cos\theta$ makes this integral depend on the integral

$$f_{2m}(X)=\int_0^X 2e^{-x^2}x^{2m}dx,$$

which satisfies the recurrence relation

$$(2m+1)f_{2m}-2f_{2m+2}=2e^{-X^2}X^{2m+1}$$

and can be expressed in the form

$$f_{2m}(X)=2X^{2m+1}e^{-X^2}S_m(X^2),$$

where

$$S_m(x)=1/(2m+1)+x/(2m+1)(2m+3)+\cdots.$$

The first integral is thus $2e^{-X^2}S_{n-p+p_1}(X^2)$. Representations are also found for a function $F_\nu^{(0)}$ given by a double integral. It is necessary to study the function

$$I_p(x)=x^{2m+2p-1}(m+p)!/[m!(2m+2p)!],$$

which satisfies $dI_p/dx=\frac{1}{2}xI_{p-1}$ and is of the form $P_p(x)e^x+Q_p(x)e^{-x}$, where P_p, Q_p are polynomials whose degrees do not exceed $p-1$. Other integrals are considered.

H. Bateman (Pasadena, Calif.).

Oseen, C. W. Contributions à la théorie analytique des marées. III. Ark. Mat. Astr. Fys. **29A**, no. 1, 37 pp. (1943). [MF 12009]

[Cf. the preceding review.] The author rediscovered, as a special case, the solution which formed the starting point

of Ekman's theory of ocean currents on a rotating earth. In his formulae $\nu \pm 2\Omega$ plays the same part as the quantity $1/c = 2\omega \sin \phi$ in Ekman's theory. The distribution of velocity in the ocean depends not only upon the earth's rotation but also upon the oscillations of the ocean. Ekman's theory is generalized still further by supposing that the pressure is a linear function of the coordinates of a point in a plane perpendicular to the axis of rotation. Ekman considered constant pressure and no body forces. *H. Bateman.*

Oseen, C. W. Contributions à la théorie analytique des marées. IV. Ark. Mat. Astr. Fys. **29A**, no. 8, 10 pp. (1943). [MF 12011]

A further study is made of the solution in polar coordinates and Bessel functions found in part III [cf. the preceding review]. A new representation is given for the function

$$F_{m,n}^{(p)} = (2\pi)^{-1} \int_0^{2\pi} e^{-(p\theta - n\psi)} F_{m,n} d\theta,$$

where

$$F_{m,n} = c_1 Z_n(A_1 r) + c_2 Z_n(A_2 r) + c_3 Z_n(A_3 r) \exp\,(in\psi + im(\Omega/\mu')^{\frac{1}{2}} x_3)$$

and $A_s = (-a_s/\mu')^{\frac{1}{2}}$ $(s = 1, 2, 3)$, c_1, c_2, c_3 being arbitrary constants and a_1, a_2, a_3 roots of a cubic equation. The series found has rather complicated coefficients $C_{r,\lambda}^{n,p}$ which satisfy the relation $C_{r,\lambda}^{n,p} = (-1)^n C_{r,\lambda}^{-n,p}$. *H. Bateman.*

Oseen, C. W. Contributions à la théorie analytique des marées. V. Ark. Mat. Astr. Fys. **29A**, no. 9, 9 pp. (1943). [MF 12012]

[Cf. the three preceding reviews.] The aim is to study the simplest solutions of the equations of tidal oscillations in a rotating system which can be expressed in terms of known analytical functions. The first step is to form solutions of

$$(\mu'\Delta - i\nu)^2 \Delta F + 4\Omega^2 \partial^2 F/\partial x_3^2 = 0$$

of the form

$$F = f_m(x_1, x_2) \exp\,\{im(\Omega'/\mu)^{\frac{1}{2}} x_3\},$$

where m is a nondimensional number. The equation for f is

$$(\mu'\Delta_2 - a_1)(\mu'\Delta_2 - a_2)(\mu'\Delta_2 - a_3) f = 0,$$

where

$$\Delta = \Delta_2 - \Omega m^2/\mu', \quad \Delta_2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2,$$

and a_1, a_2, a_3 are the roots of a cubic equation. Use is made of the substitution

$$f_m = \sum_{n=-\infty}^{\infty} f_{m,n}(r) e^{in\psi}, \quad x_1 = r\cos\psi, \quad x_2 = r\sin\psi,$$

and $f_{m,n}$ is expressed as a Bessel function. Use is made also of the substitution $r = aR_{a,b}\sin\theta$, $x_3 = bR_{a,b}\cos\theta$, particularly when $a = b = 1$, when $R_{a,b}$ is replaced by R. A study is made of the partial differential equation $(\mu'\Delta - i\nu)^2 F = f(R)$, which leads to some interesting analytical developments. *H. Bateman* (Pasadena, Calif.).

Landau, L. On the problem of turbulence. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 311–314 (1944). [MF 12041]

The author considers the problem of turbulence with the idea that the problem may appear in a new light if the process of initiation of turbulence is examined thoroughly. He considers discrete normal modes of oscillation superposed on the principal motion. When the Reynolds number R is increased slightly beyond the critical Reynolds number R_c of the first normal mode of angular frequency ω_1, this mode of oscillation is built up to an amplitude proportional to $(R - R_c)^{\frac{1}{2}}$. With further increase of the Reynolds number, the flow develops into a general periodic motion with angular frequency ω_1 and an unspecified phase. The author considers this motion in turn as the principal motion and studies its stability problem. By continuing this process, the author regards turbulent motion as a quasi-periodic motion. *C. C. Lin* (Pasadena, Calif.).

Sedov, L. I. Decay of isotropic turbulent motions of incompressible fluid. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 116–119 (1944). [MF 11628]

The author reviews and discusses the problem of isotropic turbulence based on the von Kármán-Howarth equation for the propagation of the correlation function. It is pointed out that Loytzansky's invariant integral [M. D. Millionshtchikov, same C. R. (N.S.) **22**, 231–235 (1939)] takes a finite value for only one member of the family of correlation functions obtained by von Kármán and Howarth for low Reynolds number of turbulence. For strictly self-preserving correlation functions, the author obtains a law of decay of turbulence, which in general differs from both the half-power law given by Dryden [Quart. Appl. Math. **1**, 7–42 (1943); these Rev. **4**, 263] and the five-seventhspower law given by Loytzansky. It is shown that, unless the decay of turbulence follows the half-power law, strict self-preservation of the correlation functions requires the correlation function of the second order to take the general shape given by von Kármán and Howarth for low Reynolds numbers. For this general case, the triple correlation function is given explicitly in terms of the double correlation function. [Reviewer's remark. Experiments of S. Corrsin at California Institute of Technology give a power law of decay with a power between $\frac{1}{2}$ and 5/7. According to his experimental results, the rate of increase of scale is several hundred times that accountable by viscosity. It is to be noted that in the author's paper the shape of the correlation functions is determined through such viscosity terms. Note also that the work of the present author and that of Dryden depend upon strict self-preservation up to small distances between points of observation of the correlations, while Loytzansky's law depends only upon a sufficiently approximate self-preservation at large distances and the finiteness of his invariant integral.] *C. C. Lin* (Pasadena, Calif.).

Chou, P. Y. On velocity correlations and the solutions of the equations of turbulent fluctuation. Quart. Appl. Math. **3**, 38–54 (1945). [MF 12084]

The author continues his study of the solution of the equations of turbulent fluctuation by the construction of the correlation functions [Chinese J. Phys. **4**, 1–33 (1940); these Rev. **3**, 285]. Equations governing correlation functions of second and third orders are derived and studied in detail. The pressure-velocity correlations in these equations are transformed into velocity correlations at two points by solving Poisson's equation for pressure. This process reduces the turbulence problem to the solution of a set of nonlinear partial integrodifferential equations. After suitable transformations and approximations, the author finds it possible to put the final equations into forms suitable for the calculation of the distribution of the mean velocity and the correlation functions. The initial approximation uses only the original equations of Reynolds. *C. C. Lin.*

Kolscher, M. Discontinuous solutions of the equations of motion of fluid flow. Ministry of Aircraft Production [London], R. T. P. Translation no. 2403, 15 pp. (1945). [MF 12621]

[Reprinted by the Durand Reprinting Committee.] The original appeared in Luftfahrtforschung 17, 154–160 (1940); cf. these Rev. 4, 59.

Ishlinsky, A. J. A slow motion of a viscous fluid in a tube of variable cross-section. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 395–400 (1944). (Russian. English summary) [MF 12390]

The problem is considered for a tube of circular cross-section having a variable radius $r=r_0+\delta \sin az$, where δ is a very small value in comparison with r_0. The inertia terms in the equations for the motion of an incompressible viscous fluid are not taken into consideration. For the components u_z and u_r of the velocity of fluid along the axis of the tube and along the radius the formulae

$$u_z=U(1-r^2/r_0^2)+(a/r)(Au'+C(2u'+xu))\sin az,$$

$$u_r=-r^{-1}\partial\chi/\partial z=-(a/r)\varphi(ar)\cos az$$
$$=-(a/r)(Au+Cxu')\cos az$$

are obtained, where $x=ar$, $u=-ixJ_1(ix)$ and U is the velocity of the fluid on the axis of the tube. The constant values A and C are determined by

$$A=\frac{-2xuu'U\delta}{a(2uu'+xu^2-xu'^2)},\qquad C=\frac{2uU\delta}{a(2uu'+xu^2-xu'^2)},$$

where $x=ar_0$. The formula for the distribution of pressure p is

$$p=p_0-\frac{8\mu Ua^2\delta J_0(ix)\cos az}{x_0[x_0^2J_1^2(ix_0)-2J_0(ix_0)J_2(ix_0)]},\qquad x_0=ar_0.$$

From the author's summary.

Seth, B. R. On the stress-strain velocity relations in equations of viscous flow. Proc. Indian Acad. Sci., Sect. A. 20, 329–335 (1944). [MF 12201]

The author considers nonlinear stress-strain velocity relations (including quadratic terms in the strain velocities) for the flow of viscous fluids. The following examples (in steady state motion) are studied: (1) flow of fluid between two parallel fixed plates; (2) flow of fluid through a pipe; (3) two-dimensional simple rotary motion (the case of liquid rotating between two co-axial cylinders). The author is able to obtain, from physical considerations, solutions which satisfy the equations of motion, the continuity equation and the boundary conditions. In each of the first two cases, the author shows that the minimum pressure will be negative (instability of flow) unless some bound is fixed for the velocity. In the third example, the motion is always possible if the inner cylinder is fixed. *N. Coburn* (Austin, Tex.).

Jung, F. Über den Zusammenhang einiger Formeln für den hydrodynamischen Auftrieb. Z. Angew. Math. Mech. 21, 108–114 (1941). [MF 12152]

The author derives two general formulae for the force and moment acting on a surface which contains stream lines, in a steady three-dimensional rotational flow of an incompressible nonviscous fluid. The general formulae are reduced in different special cases to well-known results. For the two-dimensional case, the Blasius formulae are obtained. *H. S. Tsien* (Pasadena, Calif.).

Mangler, Werner. Die "ähnlichen" Lösungen der Prandtlschen Grenzschichtgleichungen. Z. Angew. Math. Mech. 23, 241–251 (1943). [MF 11737]

The author studies a flow along one face of a wedge. The motion is two-dimensional and is reinterpreted as the flow along a ray which forms one side of an angle. It is assumed that near the ray the flow satisfies the Prandtl boundary layer equations whereas outside the boundary layer the flow is analytic. The Reynolds number is assumed to be large but not so large as to produce turbulence. On the ray the velocity is zero and on the outer perimeter of the boundary layer the velocity is a function $U(x)$, where x is measured parallel to the ray. Within the boundary layer the component $u(x, y)$ of the velocity parallel to the ray is required to satisfy the similarity condition

$$u(x+\Delta x, \lambda y)/U(x+\Delta x)=u(x, y)/U(x).$$

It is easy to find a complex velocity potential which will produce a $U(x)$ satisfying this restriction. By studying the conformal mapping associated with this potential it is found that the motion is the desired flow along a face of a wedge. The latter part of the article contains a study of the manner in which the boundary layer flow unites with the analytic flow. *A. H. Copeland* (Ann Arbor, Mich.).

Flancl, P. I., and Eterman, I. I. The calculation of the flow past a body approaching an oblong ellipsoid of revolution. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 65–69 (1944). (Russian. English summary) [MF 11469]

Suppose that a given surface A is close to a surface of revolution B. Let φ be the solution of the external Neumann problem for A and let n denote the external normal to B, for which Neumann's problem is assumed to possess an explicit solution. It is shown that the values of $d\varphi/dn$ on the surface B can be found approximately without solving Neumann's problem for A. Thus Neumann's problem for A can be reduced to a certain extent to the same problem for B. *A. Weinstein* (Toronto, Ont.).

Westberg, Ragnar. The forces on a body in an incompressible viscous fluid. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 14, no. 1, 15 pp. (1944). [MF 11550]

The author derives general formulae giving the resultant force and the resultant moment of force acting on a body moving in an incompressible viscous fluid which is subjected to body forces. All the integrals in those formulae which depend on the motion of the fluid are transformed, as far as possible, into surface integrals involving kinematic quantities over an arbitrary surface. This is always possible in the case of the force; it is possible for the moment of force only under certain restrictions. The cases of irrotational motion and two-dimensional motion are discussed in greater detail. *C. C. Lin* (Pasadena, Calif.).

Slezkin, N. A. Penetration of a thin plate into a viscous medium. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 7–9 (1945). [MF 12545]

Serebriisky, J. M. Particular solutions of non-linear equation in the theory of the wing of finite span. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 267–272 (1944). (Russian. English summary) [MF 12226]

The author considers particular solutions of Prandtl's equation for a wing having an elliptic form when the lift

coefficient depends nonlinearly on the incidence. The methods used involve the use of the trigonometric substitutions of Glauert and equating to zero the coefficients of the Fourier expansions of certain identically vanishing functions. It is shown that for certain lift coefficient curves there is a value of the incidence (larger than the critical angle) starting from which a real solution of Prandtl's equation does not exist. For simple parabolic relationship between the lift coefficient and the angle of attack the author obtains formulae for transition from the basic relationships for an airfoil to the relationships for a wing of finite span.

H. P. Thielman (Ames, Iowa).

Müller, Wilhelm. Über die Berechnung der Kräfte an einem in der reibungslosen Flüssigkeit beschleunigt bewegten Tragflügel. Z. Angew. Math. Mech. 23, 305–320 (1943). [MF 11748]

The author applies his general formulae developed in a previous paper [Ing.-Arch. 14, 332–350 (1944); these Rev. 6, 192] to obtain the forces and moments acting on a two-dimensional Joukowski airfoil moving in a general manner in an inviscid fluid. The method of complex variables is used. [Reviewer's remark. As there seem to be inconsistencies in the complex form of the general moment formula, the results obtained for moments should be regarded with reserve.]

C. C. Lin (Pasadena, Calif.).

von Kármán, Theodore, and Tsien, Hsue-Shen. Lifting-line theory for a wing in non-uniform flow. Quart. Appl. Math. 3, 1–11 (1945). [MF 12081]

The authors study a lifting line placed in a parallel stream whose velocity is perpendicular to the span and is assumed to vary in both directions normal to the flow. They take the x-axis downstream, y-axis along the span and z-axis vertically upward; the undisturbed flow velocity is $U(y, z)$; the total velocity components are $U+u$, v and w. The equations of motion are linearized on the assumption that $U \gg u, v, w$; that is, the lifting line is lightly loaded. The resulting equation for the pressure p is

$$\frac{1}{U^2}\frac{\partial^2 p}{\partial x^2}+\frac{\partial}{\partial y}\left(\frac{1}{U^2}\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{U^2}\frac{\partial p}{\partial z}\right)=0.$$

The boundary conditions are the vanishing of p at infinity and the requirement that the lift on the lifting line is equal to that on the wing, $l(y)$. The solution can be built up of elementary solutions of the form $P(y, z, \lambda)\cos \lambda x$, using the Fourier integral. The differential equation and boundary conditions for $P(y, z, \lambda)$ are set up, and the induced velocities u, v, w are calculated in the form of Fourier integrals. The apparent difficulty that $u(x, y, z)\to\infty$ as $x\to\infty$ is discussed.

The authors consider conditions in the "Trefftz plane," far downstream, normal to the x-axis. For the velocity components in this plane it is found that

$$v(\infty, y, z)=\varphi_z/(\rho U), \quad w(\infty, y, z)=\varphi_y/(\rho U),$$

where ρ is the fluid density and $\phi(y, z)=-P(y, z, 0)$ satisfies

$$\frac{\partial}{\partial y}\left(\frac{1}{U^2}\frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{U^2}\frac{\partial \phi}{\partial z}\right)=0;$$

this is actually the equation of continuity. The corresponding modified vorticity equation

$$\partial(Uv)/\partial z-\partial(Uw)/\partial y=0$$

applies for all values of x. The induced drag can be evaluated by an integral involving ϕ and $\partial\phi/\partial z$ in the Trefftz plane; the minimum of this integral, the total lift being held constant, occurs when the downwash angle w/U at the lifting line is constant. The case $U=U(y)$ is treated separately. Here

$$\phi(y, z)=\int_0^\infty f(\lambda)e^{-\lambda z}Y_\lambda(y)d\lambda,$$

where the differential equation for $Y_\lambda(y)$ is

$$Y_\lambda''-2(U'/U)Y_\lambda'+\lambda^2 Y_\lambda=0$$

and $f(\lambda)$ is to be determined by an integral equation involving $l(y)$ or the geometrical characteristics of the wing.

W. R. Sears (Inglewood, Calif.).

Kucharski, W. Eine Integralgleichung für den rotierenden Schaufelstern und ihre Lösung. Z. Angew. Math. Mech. 21, 65–79 (1941). [MF 12146]

Consider incompressible flow in a two-dimensional radial vane system rotating with a constant angular velocity ω. The n vanes are assumed to be straight line segments of length l spaced at regular angular intervals of $2\pi/n$. The author solves the problem by considerations which are often used in airfoil theory. Let $m(\xi)$ be a distribution of vorticity along one vane. This distribution must be such that the normal velocity at any distance x along the vane vanishes. This leads to the integral equation

$$\int_0^l m(\xi)/(x^n-\xi^n)d\xi=(\omega/n)x^{-n+2}. \tag{1}$$

The equivalent of the Kutta condition requires that $m(l)=0$. By a simple transformation (1) becomes

$$\int_0^\pi \frac{Y(\psi)}{\cos\varphi-\cos\psi}d\psi=-\frac{\omega}{2\pi l^{n-2}\sin^{(n-2)/n}\frac{1}{2}\varphi}. \tag{2}$$

With

$$Y(\psi)=\tfrac{1}{2}b_0+\sum b_\nu\cos\nu\psi,$$

one obtains from (2)

$$\sum b_\nu \sin\nu\varphi=\frac{\omega\sin\varphi}{\pi l^{n-2}\sin^{(n-2)/n}\frac{1}{2}\varphi}. \tag{3}$$

If $n\leqq 4$ the b_ν can be obtained from (3). If $n>4$ the right side of (3) becomes infinite at $\varphi=0$. The author obtains a solution for $n>4$ by writing the kernel of the integral equation as a sum of two functions, one of which is chosen so that the singularity is removed. The other function is then again obtained by means of Fourier series. The solution is applicable to flow in radial compressors if a suitable source solution is superimposed. The case of slightly cambered vanes is also discussed; the methods and results here are quite similar to the thin airfoil theory. Explicit solutions are given for the vorticity distribution, the circulation, etc.

H. W. Liepmann (Pasadena, Calif.).

Pinl, M. Zur geometrischen Deutung und Transformation der Grundgleichung der ebenen kompressiblen Potentialströmung. Z. Angew. Math. Mech. 21, 80–85 (1941). [MF 12151]

The familiar velocity potential equation for a plane compressible potential flow is known to be the Euler equation of the variational problem for $\iint(1+p^2+q^2)^{\frac{1}{2}}dA$, subject to $\iint\{\lambda+\mu(p^2+q^2)\}^{\gamma}dA=C$, where C, λ, μ are positive constants and the other symbols are standard. This is of the form of a generalized relative minimal surface problem and accordingly the solutions are expected to correspond to vanishing of an associated mean curvature. The curvature tensor

is determined by the author. Some canonical forms for the Euler equation are given using the standard parametric curves introduced in classical differential geometry for two-dimensional surfaces (isotropic or isothermal parameters).
D. G. Bourgin (Urbana, Ill.).

Fortier, André. Sur la formule de Bernoulli généralisée. C. R. Acad. Sci. Paris **218**, 384–386 (1944). [MF 12111]

The author states that the increase in mechanical power Δw in a real incompressible fluid (steady state), bounded by two sections S_1, S_2 of a conductor, may be obtained from the generalized Bernoulli equation

$$(1)\quad \Delta w=\int\int_{S_2}(\tfrac{1}{2}\rho V^2+p+U)v_n d\sigma - \int\int_{S_1}(\tfrac{1}{2}\rho V^2+p+U)v_n d\sigma,$$

where ρ is the density, V the velocity, p the pressure, U the potential of the external forces and v_n the projection of the velocity on the surface normal to the element $d\sigma$. Evidently, if the fluid is perfect, $\Delta w=0$. It is the author's purpose to generalize (1) for the case of a compressible fluid. He outlines a proof by energy considerations that, for a fluid of volume W bounded by a surface S,

$$dE/dt+\Delta w=-\int\int_S(\mathbf{T}-p\mathbf{n}+\tfrac{1}{2}\rho V^2\mathbf{n}+\rho P\mathbf{n}+\rho U\mathbf{n})\cdot\mathbf{V}d\sigma,$$

where

$$(2)\quad E=\int\int\int_W\rho(\tfrac{1}{2}V^2+P-p/\rho+U)d\tau,\quad P=\int dp/\rho,$$

and $\mathbf{T}$ is the surface force, $\mathbf{n}$ the normal vector to S, $\mathbf{V}$ the velocity of the fluid, $\Delta w=dQ'/dt-dQ/dt$, dQ is the external heat added to the fluid and dQ' the internal heat generated in the fluid in time dt. By taking means of (2), the author finds, for the fluid bounded by two sections S_1, S_2 of a conductor,

$$\overline{\Delta w}=\int\int_{S_1}\overline{(\mathbf{T}-p\mathbf{n}+\tfrac{1}{2}\rho V^2\mathbf{n}+\rho P\mathbf{n}+\rho U\mathbf{n})\cdot\mathbf{V}d\sigma} - \int\int_{S_2}\overline{(\mathbf{T}-p\mathbf{n}+\tfrac{1}{2}\rho V^2\mathbf{n}+\rho P\mathbf{n}+\rho U\mathbf{n})\cdot\mathbf{V}d\sigma}.$$

This is the generalization of (1) which the author seeks.
N. Coburn (Austin, Tex.).

Hantzsche, W. Die Prandtl-Glauertsche Näherung als Grundlage für ein Iterationsverfahren zur Berechnung kompressibler Unterschallströmungen. Z. Angew. Math. Mech. **23**, 185–199 (1943). [MF 11744]

The author applies an iteration procedure to the study of a two-dimensional adiabatic irrotational flow of a compressible fluid about a profile. He assumes that the stream function is expansible in a power series in a parameter λ with coefficients which are functions of the two spatial coordinates. The partial sums of this series constitute the successive approximations to the flow. The stream function is substituted in the Prandtl-Glauert equation and the method of equating coefficients is applied. This produces a series of differential equations which together with the boundary conditions determine successively the various coefficients in the development of the stream function. These equations are converted into an equivalent system involving the partial sums. The first of these latter equations gives the Prandtl-Busemann approximation to the stream function. The second was previously studied by the author and H. Wendt. The third is the concern of the present paper.

A stretching transformation converts the equations into a form in which the Laplacian of a partial sum is expressed in terms of partial sums determined by previous equations. A conformal mapping converts the profile into a circle. An additional complex transformation reduces the problem of solving the equations to quadratures. The author applies his method to two examples. *A. H. Copeland.*

Tsien, Hsue-Shen, and Lees, Lester. The Glauert-Prandtl approximation for subsonic flows of a compressible fluid. J. Aeronaut. Sci. **12**, 173–187, 202 (1945). [MF 12217]

The content of this paper is similar to that of a paper by Goldstein and Young [Ministry of Aircraft Production [London], Aeronaut. Res. Committee, Rep. and Memoranda no. **1909** (6865), 1–20 (1943); these Rev. **6**, 193]. The cases treated are two- and three-dimensional wind-tunnel blockage, two-dimensional wind-tunnel wall interference on a thin airfoil, three-dimensional wing theory and downwash at a tail surface, three-dimensional wind-tunnel wall interference on a wing, and the theory of lightly loaded propellers. A derivation is included of the so-called Biot-Savart formula (giving the velocity induced by a vortex filament) according to the linearized theory. In all cases the transformation employed is the one designated "II" by Goldstein and Young.

In calculating two-dimensional wind-tunnel corrections, the present authors use a more accurate method and show, for comparison, the results of the simplified analysis. On the other hand, their formulas do not include corrections for drag or pitching moment, while these are usually given in the other work. In all comparable cases the results of the two papers are in good agreement, excepting the case of two-dimensional wind-tunnel interference on an airfoil, where Goldstein and Young's formula implies a more rapid increase of lift coefficient with Mach number than is exhibited by the approximate (supposedly equivalent) results of this paper. *W. R. Sears* (Inglewood, Calif.).

Tatarenchik, V. S. Concerning particular solutions for equations of gas dynamics. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 401–412 (1944). (Russian. English summary) [MF 12391]

The author deals with some simple flows in a compressible fluid corresponding to the exact solutions for the equations of gas dynamics and analogous to those in an incompressible fluid. On the basis of these examples of fluid motion the passing of the flow beyond the velocity of sound can be considered as well as the further conduct of the flow at super-sound velocities. The analogy with the motions of a noncompressible fluid gives the possibility of establishing the change in the structure of a flow due to an increase in the velocity. *Author's summary.*

Kiebel, I. A. Application of the method of long waves to a compressible fluid. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 413–416 (1944). (Russian. English summary) [MF 12392]

In the case of a stationary motion of a heavy barotropic (in particular, incompressible) fluid the long-wave method gives the pressure in functions of the height by means of simple quadratures. For the barocline fluid the method of long waves cannot be applied directly. The author first shows how the equations of hydrodynamics must be trans-

formed, after which the pressure is determined by an ordinary differential equation of the second order. The author then shows how this equation can be solved in the problem of flow past mountain chains and in certain questions of gas dynamics. *Author's summary.*

Haskind, M. D. Oscillations of a system of plates on the surface of a heavy liquid. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 421–430 (1943). (Russian. English summary) [MF 11262]

The author considers the wave motion in two dimensions due to small oscillations of a system of plates on the surface of a heavy fluid of infinite depth. The complex potential $W(z, t)=w(z)\exp(ikt)$ satisfies the conditions $\Im(dw/ds+i\nu w)=0$, $\nu=k^2/g$ on the free parts of the boundary. The same quantity takes opposite values on the two borders of the segments representing the plates. Using Schwarz's reflection principle and Cauchy's theorem, the author obtains an integral equation for the velocity potential; this equation is discussed for small values of ν.

A. Weinstein (Toronto, Ont.).

Jones, R. Clark. On the theory of the directional patterns of continuous source distributions on a plane surface. J. Acoust. Soc. Amer. **16**, 147–171 (1945). [MF 11735]

Let $z=0$ be a plane executing vibrations with velocity $A(x, y)\exp i\omega t$. The sound pressure $P(R, \theta, \phi)$, in the medium of density ρ, is given formally by

$$\frac{i\omega\rho\exp i(\omega t-kR)}{2\pi R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(x, y)\times\exp(ik\sin\theta(x\cos\phi+y\sin\phi))dxdy.$$

It is assumed that $A(x, y)$ is so restricted that the Fourier double integral, denoted by $Q(\theta, \phi)$, has meaning. The "practical problem" is that of determining a vibration pattern $A(x, y)$ to yield a desired directional distribution $Q(\theta, \phi)$.

In principle this is elementary (for Q and A are obviously Fourier integral transforms). For applications in technology the writer's procedure is to find the explicit solution in closed form when A has circular symmetry or is a line distribution. Using the practical criterion that the sources must be of some given size, he determines the best approximating $A(x, y)$ having the symmetries mentioned above, uniform phase (that is, A is real) and simple functional form. [For instance, for the circular case he uses $A=(1-(r/a)^{3m})^r$.] These distributions are superposed. Extensive numerical tables are included.

D. G. Bourgin (Urbana, Ill.).

Theory of Elasticity

Thomas, T. Y. Maximum angular variation under small displacements. Amer. Math. Monthly **52**, 73–81 (1945). [MF 11905]

The author considers the change in angle between two unit vectors in a three-dimensional Euclidean space under the point displacement of elasticity theory. It is shown that a maximum or minimum value of the variation in the angle between the two vectors must exist, and that this variation is stationary only when the vectors are perpendicular, unless there is no variation for two arbitrary vectors. The author furnishes a detailed discussion for the determination of those directions for which the variation in angle is stationary. This discussion is based upon a classification of the strain tensor according to whether one, two or three distinct principal strains exist. In each case, the desired directions and the variation in angle are explicitly determined; two of these stationary values furnish maximum and minimum variations for the angle. *N. Coburn* (Austin, Tex.).

Krivoshein, Nicolás. Generalization of the Mohr circle and the Lamé ellipse. Revista Union Mat. Argentina **10**, 69–77 (1945). (Spanish. English summary) [MF 12276]

Teodoresco, C. C. Sur la déformation autour d'un point. Mathematica, Timişoara **20**, 65–68 (1944). [MF 12470]

Vlasov, V. Z. The equations of continuity of deformations in curvilinear coordinates. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 301–306 (1944). (Russian. English summary) [MF 12229]

The author uses tensor methods to deduce the equations of compatibility for a general curvilinear coordinate system. The strain tensor e_{ij} is defined by the relation $\delta g_{ij}=e_{ij}$, where g_{ij} is the metric tensor and δ denotes the variation in passing from the unstrained state to the strained state. Substitution from this equation in the relation $\delta R_{ijkl}=0$ (R_{ijkl} is the familiar curvature tensor) yields the desired equations of compatibility. *G. E. Hay* (Providence, R. I.).

Shermann, D. I. A spatial static problem of the elasticity theory with given displacement on the boundary. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 341–360 (1943). (Russian. English summary) [MF 11237]

Lauricella's method for construction of integral equations [Nuovo Cimento (5) **13**, 104–118, 155–174, 237–262, 501–518 (1907)] is extended to the case of bodies bounded by several surfaces. *A. Weinstein* (Toronto, Ont.).

Scherman, D. J. Une nouvelle solution du problème plan de la théorie de l'élasticité pour un milieu anisotrope. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 314–315 (1941). [MF 10988]

The author gives a new method for solving the plane problem of elasticity for an anisotropic medium. The stresses can be expressed in terms of two analytic functions of a complex variable, which satisfy certain boundary conditions. The determination of these functions can be reduced to the solution of a system of integral equations of the Fredholm type which have a unique solution. *S. Bergman.*

Rachevsky, P. Sur l'équilibre des corps élastiques possédants une symétrie hélicoïdale. Rec. Math. [Mat. Sbornik] N.S. **15** (**57**), 55–70 (1944). (Russian. French summary) [MF 12281]

The author considers elastic bodies having helical symmetry in both the unstrained and strained states. Because of these symmetry conditions, the displacements cannot be general functions of three variables but must have more restricted functional forms. When these are substituted into the equations of equilibrium, it is found that the problem reduces to the determination of three harmonic functions possessing helical symmetry. *G. E. Hay.*

Urquijo Mercado, Miguel. Three studies on the semi-infinite elastic body. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario 1943, 51–57 (1944). (Spanish) [MF 12346]

The author discusses two problems. (1) Given a semi-infinite elastic body under arbitrarily distributed loads acting upon its limiting plane; find the displacement at any point of this plane. The author discusses the well-known solution of this problem given by H. Hertz and suggests a new method for the evaluation of the integrals involved in this solution based on a combination of numerical work and the use of a polar planimeter.

Problem (2) is the inverse of (1). Boussinesq's solution of this problem is studied and an apparatus for its numerical evaluation suggested. The apparatus is essentially a combination of Peaucellier's inversor and a polar planimeter. The third study deals with a special case of the problem discussed in the second; a numerical method is suggested for its solution. *P. Neményi* (Pullman, Wash.).

Seth, B. R. Consistency equations of finite strain. Proc. Indian Acad. Sci., Sect. A. 20, 336–339 (1944). [MF 12202]

The author considers the case of nonlinear strain (that is, the case where the strain tensor contains quadratic terms in the derivatives of the displacement vector). The problem is to determine the consistency relations. This is accomplished by determining the metric tensor for the strained Euclidean space and requiring that the Riemann-Christoffel tensor of this space vanish. The vanishing of the six independent components of this tensor furnishes the required relations. By use of the previously determined metric tensor, explicit formulas are obtained. *N. Coburn.*

Ishlinsky, A. J. Some applications of statistics in describing deformation laws. C. R. (Doklady) Acad. Sci. URSS (N.S.) 45, 179–180 (1944). [MF 12579]

The author discusses the (one-dimensional) stress-strain relations of certain materials which are composed of an infinite number of elements. All elements obey the same type of simple stress-strain relation, but the constants appearing in this relation may assume different numerical values for the various elements. The composite material is then characterized by the elementary stress-strain relation and by the distributions of the numerical values of the constants over the elements. Typical cases of hysteresis and after-effect are discussed. *W. Prager* (Providence, R. I.).

Weiskopf, Walter H. Stresses in soils under a foundation. J. Franklin Inst. 239, 445–465 (1945). [MF 12539]

Soil is assumed to have a compression modulus E and a shear modulus G. Because of the slipping of granules on each other, the resistance to shear is much less than in the idealized elastic solid. Thus the quantity E/G is a soil constant which varies for different soils. Both two-dimensional and three-dimensional problems are considered. The approach resembles the familiar one used for an ordinary semi-infinite elastic body with a plane boundary to which loads are applied. A stress function is introduced and is found to satisfy a partial differential equation similar to the biharmonic equation, but involving the soil constant E/G and Poisson's ratio. Solutions are written down, and it is found that they can be made to satisfy all the boundary conditions corresponding to a concentrated normal load. Integration then gives the solution in the case of a distributed normal load. The analytic solutions are found to agree rather well with empirical data. *G. E. Hay.*

Harding, J. W., and Sneddon, I. N. The elastic stresses produced by the indentation of the plane surface of a semi-infinite elastic solid by a rigid punch. Proc. Cambridge Philos. Soc. 41, 16–26 (1945). [MF 12372]

The authors consider the elastic problem arising when a solid rigid punch is pressed against the plane boundary of a semi-infinite elastic solid. The punch has an axis of symmetry perpendicular to the plane boundary. By use of Hankel transforms, the problem is reduced to a pair of dual integral equations of a type for which the solution is known. The method is applied to the cases when the punch is (i) conical, (ii) a rigid sphere, (iii) a flat-ended circular cylinder. The first case has been considered previously by A. E. H. Love, using a different method, and his results are found to agree with those of the present paper.

G. E. Hay (Providence, R. I.).

Shapiro, G. S. Equilibrium of a cone and a conic shell. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 332–336 (1944). (Russian. English summary) [MF 12233]

The author considers axially symmetric deformations of a cone and of a conical shell under the assumption that the load on the boundary (that is, for $\theta=\alpha$ in the case of a cone and for $\theta=\alpha_1$, $\theta=\alpha_2$ in the case of a shell) is a polynomial in r (r, θ, ϕ denote polar coordinates). The stresses and deformations can be expressed in terms of a stress function (analogous to that introduced by Love and Galerkin), which satisfies the biharmonic equation. Writing

$$\varphi=\sum_{n=3}^{\infty} r^n(A_nP_n+B_{n-3}P_{n-2}+C_{n-3}Q_n+D_{n-3}Q_{n-2}),$$

where the P_n and Q_n are, respectively, Legendre polynomials and Legendre functions of the second kind, the author indicates how to determine the A_n, B_{n-3}, C_{n-3}, D_{n-3} from the given load. Several special cases (constant load, a cone under its own weight, etc.) are discussed in detail.

S. Bergman (Providence, R. I.).

Goldenveiser, A. L. Applicability of the general theorems of elasticity to thin shells. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 3–14 (1944). (Russian. English summary) [MF 11465]

The classical equations for the equilibrium of thin shells are deduced from the general equations of elasticity by using certain assumptions. The author investigates whether some general theorems of elasticity (such as Betti's reciprocity law and Kirchhoff's unicity theorem) are still valid in the approximate theory. It is shown that Betti's and Kirchhoff's theorems are valid when the stress-strain relations are given by linear equations with a symmetric matrix and when the strain energy is nonnegative for any values of the deformation components. *A. Weinstein* (Toronto, Ont.).

Timoshenko, Stephen P. Theory of bending, torsion and buckling of thin-walled members of open cross section. J. Franklin Inst. 239, 201–219, 249–268, 343–361 (1945). [MF 12423]

In thin walled members of open cross section, it is assumed that the shearing stresses τ are uniformly distributed over the thickness t and are parallel to the tangent to the middle

line of the cross section. The shear flow is

$$\tau t=\frac{-V_y}{I_xI_y-I_{xy}^2}\left\{I_y\int_0^s ytds-I_{xy}\int_0^s xtds\right\},$$

where V_y is the resultant shear applied parallel to the y-axis, s is the arc length along the open section, and I_x, I_y and I_{xy} are components of the inertia tensor of the section. The author employs the moment of the shear flow forces to find the shear center of several types of open section beams in pure bending.

For pure torsion of thin prisms of open section an approximate expression is obtained for the displacement of the warped section, all cross sections suffering the same displacements. In the case of a cantilever beam with fixed end section, subjected to torsional couples, the sections are not all free to have the same displacements and a state of nonuniform torsion exists. The author determines an expression for the variable angle of twist depending on the state of longitudinal stresses σ_z arising in combined bending and torsion.

The author also treats the buckling of thin walled members under the action of a uniform axial end thrust which causes torsional buckling, which may or may not be accompanied by bending of the axis of shear centers. Other types of stability discussed are for bars centrally compressed and laterally supported by elastic forces which act only if buckling is impending in the direction of these forces.

D. L. Holl (Ames, Iowa).

Timoshenko, S. On the treatment of discontinuities in beam deflection problems. Quart. Appl. Math. **3**, 182 (1945). [MF 12656]

Bibliographical remarks concerning a paper by E. Kosko [same Quart. **2**, 271–272 (1944); these Rev. **6**, 139].

Djanelidze, G. J. The theory of thin-walled curvilinear bars having an unchangeable contour of the cross-section. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 25–32 (1944). (Russian. English summary) [MF 11467]

Using results of A. Lourie and of Vlasov, the author derives the equation of equilibrium and the proper boundary conditions for a curvilinear elastic thin-walled bar. The bar is considered as a surface of rotation open on both ends. Each of the boundary curves lies in a plane. The differential equation is obtained by using the principle of virtual displacement. *S. Bergman* (Providence, R. I.).

Synge, J. L. The problem of Saint Venant for a cylinder with free sides. Quart. Appl. Math. **2**, 307–317 (1945). [MF 11772]

This paper deals with the boundary value problem of the theory of elasticity for the cylinder with free sides and arbitrarily loaded ends. A considerable simplification of this problem is obtained if the requirement of arbitrarily prescribable load distributions is relinquished and certain simplifying hypotheses are introduced. The best known of these hypotheses is associated with the name of St. Venant; the author gives a new tenaorial formulation of this portion of the theory. A second less well-known hypothesis is called by the author the "exponential condition" and makes the assumption that the stress components are of the form $\sigma_{ij}=e^{hz_3}F_{ij}(x_1, x_2)$. The theory for this type of stress distributions, for the circular cylinder, has been given by J. Dougall [Trans. Roy. Soc. Edinburgh **49**, 895–978 (1914)]. The solutions of the two types of problems complement each other in a way which may be described by designating them, respectively, as "permanent free modes" and "transitory free modes." Going beyond Dougall, the author gives the formulation of the problem of the transitory free modes for the general cylinder. In this formulation he stresses the characteristic-value aspect of the problem, which leads to the three plane wave functions V, ϕ and ψ and to three simultaneous boundary conditions for these wave functions. The paper concludes with the enumeration of certain questions which the author considers worthy of future investigation. *E. Reissner* (Cambridge, Mass.).

Päsler, Max. Zur Theorie der thermischen Dämpfung in festen Körpern. Z. Phys. **122**, 357–386 (1944). [MF 11812]

The author's purpose is to take account of heat conduction in vibration problems. [This is not a new idea and, indeed, energy diffusion and viscosity as well need consideration.] Obviously $\partial/\partial t$ or dissipation terms enter. The author derives a linear damping system of equations by adding the Voight thermoelastic potential quadratic form to the ordinary potential and then using Hamilton's principle. For the transverse vibration of a bar the system can be solved. [For more than one dimension, in general, it seems that the equation system requires further added restrictions, at least if a significant perturbation method is to be used.] *D. G. Bourgin* (Urbana, Ill.).

Carrillo, Nabor. Propagation of waves in elasto-plastic media. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario **1943**, 19–35 (1944). (Spanish) [MF 12344]

The author considers simple harmonic waves under the assumption that the medium follows Hooke's law up to a certain value of shear stress and that beyond that value the shear stress increases proportionally to the time derivative of the shear strain. The total transversal wave is resolved into its elastic and plastic components according to the instantaneous local value of the shear stress. Reflection between a purely elastic and elasto-plastic medium is also considered. Longitudinal and Rayleigh waves are briefly discussed. *I. Opatowski* (Chicago, Ill.).

Ilyushin, A. A. Some problems on the theory of plastic deformations. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 245–272 (1943). (Russian. English summary) [MF 11228]

For the plastic material considered in this paper the stress tensor σ_{ij} is related to the strain tensor ϵ_{ij} by means of

$$\sigma_{ij}=K\epsilon_{kk}\delta_{ij}+2G(1-\omega)(\epsilon_{ij}-\tfrac{1}{3}\epsilon_{kk}\delta_{ij}),$$

where the bulk modulus K and the modulus of rigidity G are constants, while ω is a given function of the second invariant of the strain deviation. The differential equation for the displacement vector is established. It can be interpreted as the differential equation for the displacement vector in an elastic material which is exposed to certain fictitious surface loads and body forces in addition to the actual loads. This fact suggests a method of successive approximations in which a sequence of elastic problems are solved, the deformations obtained in one step being used to compute the fictitious loads needed in the next step. Bending of thin plates and axially symmetric deformations of thin cylindrical shells are treated as examples.

W. Prager (Providence, R. I.).

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